EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 19 February 2009
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Uted FIRST Name Connor Vol.

Lab Time Who needs lab?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may or may not find the following information useful:

$$\sum_{n=A}^{B} z^n = \begin{cases} B - A + 1 & z = 1 \\ \dfrac{z^{B+1} - z^A}{z - 1} & z \neq 1, \end{cases}$$

where A and B are integers, and $A \leq B$.

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \text{ if } |z| < 1.$$

The discrete-time unit impulse is the Kronecker delta, defined as follows:

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases}$$

Convolution Sum: The input-output relation for a discrete-time LTI system is described by the convolution sum:

$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k) = \sum_{\ell=-\infty}^{+\infty} x(\ell)h(n-\ell),$$

where x is the input and y is the output.

Frequency response of a discrete-time LTI system: If the system's impulse response is h, then its frequency response (assuming it exists), is defined as follows:

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-i\omega n}.$$

MT1.1 (20 Points) In response to every input signal x, a discrete-time system H produces a corresponding output signal y that is the *even* part of the input. That is,

$$\forall n \in \mathbb{Z}, \quad y(n) = \frac{x(n) + x(-n)}{2}.$$

For each part, explain your reasoning succinctly, but clearly and convincingly.

(a) Select the strongest true assertion from the list below.

- $\boxed{\text{(i) The system must be linear.}}$
- (ii) The system could be linear, but does not have to be.
- (iii) The system cannot be linear.

$x_1 \longrightarrow \boxed{H} \longrightarrow y_1 \qquad x_2 \longrightarrow \boxed{H} \longrightarrow y_2$

$$y_1(n) = \frac{x_1(n) + x_1(-n)}{2}$$

$$y_2(n) = \frac{x_2(n) - x_2(-n)}{2}$$

Let $x = \alpha_1 x_1 + \alpha_2 x_2 \qquad x \longrightarrow \boxed{H} \longrightarrow y$

$$y(n) = \frac{x(n) + x(-n)}{2} = \frac{\alpha_1 x_1(n) + \alpha_2 x_2(n) + \alpha_1 x_1(-n) + \alpha_2 x_2(-n)}{2} = \frac{\alpha_1[x_1(n) + x_1(-n)] + \alpha_2[x_2(n) + x_2(-n)]}{2}$$

$$= \alpha_1 \underbrace{\frac{x_1(n) + x_1(-n)}{2}}_{y_1(n)} + \alpha_2 \underbrace{\frac{x_2(n) + x_2(-n)}{2}}_{y_2(n)} = \alpha_1 y_1(n) + \alpha_2 y_2(n)$$

$\forall\, \alpha_1, \alpha_2 \in \mathbb{C}$

(b) Select the strongest true assertion from the list below.

- (i) The system must be time invariant.
- (ii) The system could be time invariant, but does not have to be.
- $\boxed{\text{(iii) The system cannot be time invariant.}}$

Let $\hat{x}(n) = x(n-N) \qquad \hat{x} \longrightarrow \boxed{H} \longrightarrow \hat{y}$

$$\hat{y}(n) = \frac{\hat{x}(n) + \hat{x}(-n)}{2} = \frac{x(n-N) + x(-n-N)}{2}$$

But

$$y(n-N) = \frac{x(n-N) + x(-(n-N))}{2} = \frac{x(n-N) + x(-n+N)}{2}$$

not equal in general

MT1.2 (20 Points) The input-output behavior of a discrete-time system is described by the following linear, constant-coefficient difference equation:

$$y(n) = \alpha y(n-1) + x(n),$$

where α is nonzero. Assume that the difference equation is solved in the forward direction in n, exactly as written above (do NOT attempt to solve it). Assume, also, that $y(-1) = 1$ is the initial condition.

For each part, explain your reasoning succinctly, but clearly and convincingly.

(a) Select the strongest true assertion from the list below.

- (i) The system must be linear.
- (ii) The system could be linear, but does not have to be.
- (iii) The system cannot be linear.

Let $x(n) = 0 \quad \forall n$. The output is still nonzero for at least one sample. In particular,

$y(0) = \alpha y(-1) + \underset{0}{\cancel{x(n)}} = \alpha \neq 0$

$y(1) = \alpha y(0) = \alpha^2$

$y(2) = \alpha y(1) = \alpha^3 \quad \ldots$

(b) Select the strongest true assertion from the list below.

- (i) The system must be time invariant.
- (ii) The system could be time invariant, but does not have to be.
- (iii) The system cannot be time invariant.

Let $\hat{x}(n) = x(n-N)$ $\qquad \hat{x} \longrightarrow \boxed{\text{System}} \longrightarrow \hat{y}$

If $\hat{y}$ is the response to $\hat{x}$, then $\hat{x}$ and $\hat{y}$ must satisfy the difference equation $\hat{y}(n) = \alpha \hat{y}(n-1) + \hat{x}(n)$ AND $\hat{y}(-1) = 1$.

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$y(n-N) = \alpha y(n-N-1) + x(n-N)$$

If $\hat{y}(n)$ is to equal $y(n-N)$, then it must be that $\hat{y}(-1) = y(-1-N) = 1$.
But we have no such constraint on y (that it be equal to 1 at $-1-N$) $\Longrightarrow$ System Not TI

MT1.3 (15 Points) Consider a discrete-time function g characterized as follows:

$$g(k) = \frac{1}{20} \sum_{n=0}^{19} e^{ik(\pi/10)n}.$$

(a) Show that

$$g(k) = \sum_{\ell=-\infty}^{+\infty} \delta(k - 20\ell),$$

where δ is the Kronecker delta function. Explain your work.

First note that g is periodic with period 20. That is

$$g(k+20) = \frac{1}{20}\sum_{n=0}^{19} e^{i(k+20)\frac{\pi}{10}n} = \frac{1}{20}\sum_{n=0}^{19} e^{ik\frac{\pi}{10}n} \underbrace{e^{i20\frac{\pi}{10}n}}_{=e^{i2\pi n}=1} = \frac{1}{20}\sum_{n=0}^{19} e^{ik\frac{\pi}{10}n} = g(k)$$

So, all we have to show now is that $g(0)=1$ and $g(k)=0$ $k=1,\cdots,19$.

Showing $g(0)=1$ is easy: $g(0) = \frac{1}{20}\sum_{n=0}^{19} e^{i0\frac{\pi}{10}n} = \frac{1}{20}\sum_{n=0}^{19} 1 = \frac{20}{20} = 1$

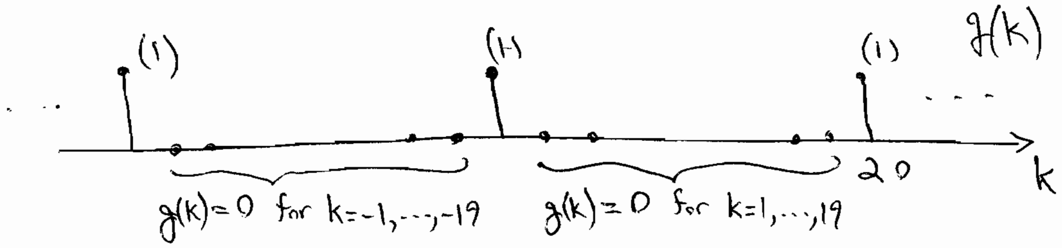
For $k=1,\cdots,19$, we know $e^{ik\frac{\pi}{10}n}$ is not identically equal to 1, so use the finite-sum formula on p. 2:

$$g(k) = \frac{1}{20}\sum_{n=0}^{19}\left(e^{ik\frac{\pi}{10}}\right)^n = \frac{1}{20}\,\frac{e^{ik\frac{\pi}{10}\cdot 20}-1}{e^{ik\frac{\pi}{10}}-1} = \frac{1}{20}\,\frac{e^{i2\pi k}-1}{e^{ik\pi/10}-1} = 0 \Rightarrow \underbrace{\begin{array}{l} g(k)=\delta(k) \quad k=0,\cdots,19 \\ g(k)=g(k+20) \end{array}}$$

$$\Downarrow$$

$$g(k) = \sum_{\ell=-\infty}^{\infty} \delta(k-20\ell)$$

(b) Provide a well-labeled plot of the function g.



periodically replicates outside the region shown

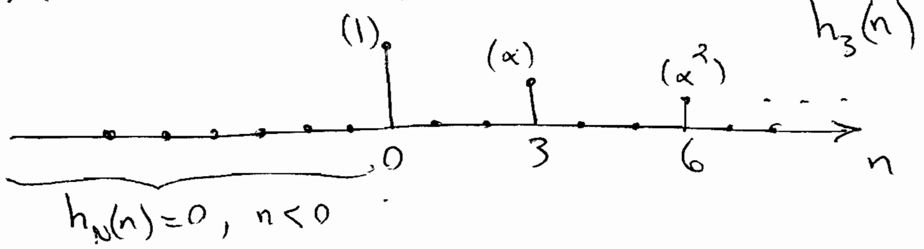
MT1.4 (25 Points) The following equation describes the impulse response of a discrete-time LTI filter:

$$\forall n \in \mathbb{Z}, \quad h_N(n) = \sum_{k=0}^{+\infty} \alpha^k \delta(n - kN),$$

where N is a positive integer, δ is the Kronecker delta function, and α is a constant parameter such that $0 < \alpha < 1$.

(a) Provide a well-labeled plot of $h_N(n)$ for $N = 3$.

$h_3(n) = \delta(n) + \alpha\delta(n-3) + \alpha^2\delta(n-6) + \cdots$



(b) Determine, in the simplest form possible, the output y of the filter in response to the input

$$x(n) = \delta(n) - \alpha\delta(n - N).$$

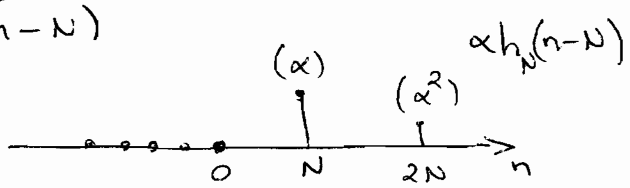
We know that $\quad x(n) \longrightarrow \underset{\text{LTI}}{\boxed{\delta(n-n_0)}} \longrightarrow y(n-n_0)$

Hence,

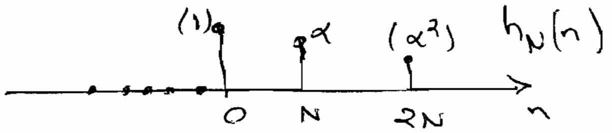
$$y_N(n) = (x * h_N)(n) = \sum_{k=0}^{\infty} \alpha^k x(n-kN) = \sum_{k=0}^{\infty} \alpha^k [\delta(n-kN) - \alpha\delta(n-kN-N)]$$

$$= \underbrace{\sum_{k=0}^{\infty} \alpha^k \delta(n-kN)}_{h_N(n)} - \alpha \underbrace{\sum_{k=0}^{\infty} \delta(n-N-kN)}_{h_N(n-N)} = h_N(n) - \alpha h_N(n-N)$$

Note $\alpha h_N(n-N)$ looks like this



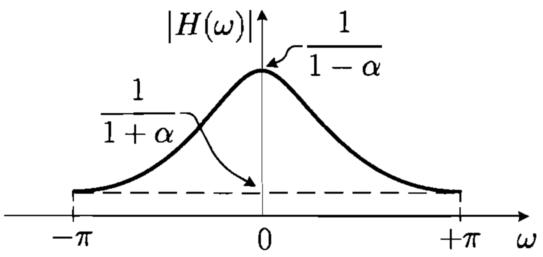
When we subtract this from $h_N(n)$



only the impulse at $n=0$ 6
remains $\Longrightarrow$ $\underline{y_N(n) = \delta(n)}$

(c) Determine a reasonably simple expression for the frequency response $H_N(\omega)$ in terms of N (use a generic positive integer N here, not a particular numerical value).

(i) For $N = 1$, we simplify the notation and omit the subscript "1" from $H_1(\omega)$. For $N = 1$, suppose the magnitude response plot $|H(\omega)|$ is as shown below,



Provide a well-labeled plot for the magnitude response $|H_2(\omega)|$. Be sure to explain your work.

For an LTI system whose impulse response g is given by $g(n) = \delta(n-N)$, we know

$$x(n) \longrightarrow \boxed{g} \longrightarrow y(n) = x(n-N)$$

and $G(\omega) = \sum_{n=-\infty}^{\infty} g(n) e^{-i\omega n} = \sum_{n=-\infty}^{\infty} \delta(n-N) e^{-i\omega n} = e^{-i\omega N} \Rightarrow G(\omega) = e^{-i\omega N}$

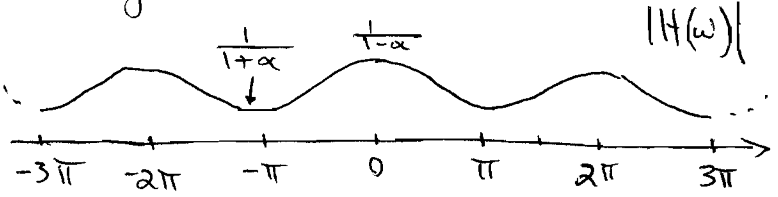
So the frequency response for H_N is straightforward:

$$H_N(\omega) = \sum_{k=0}^{\infty} \alpha^k e^{-i\omega k N} = \sum_{k=0}^{\infty} (\alpha e^{-i\omega N})^k = \frac{1}{1-\alpha e^{-i\omega N}} \Rightarrow \underline{H_N(\omega) = \frac{1}{1-\alpha e^{-i\omega N}}}$$

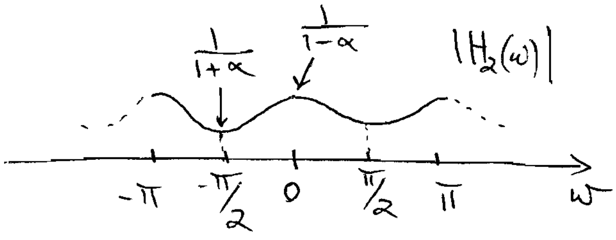
$$|\alpha e^{-i\omega N}| = |\alpha| < 1$$

$$H_1(\omega) \triangleq H(\omega) = \frac{1}{1-\alpha e^{-i\omega}} \qquad H_2(\omega) = \frac{1}{1-\alpha e^{-i\omega 2}} = H(2\omega) \Rightarrow \underline{H_2(\omega) = H(2\omega)}$$

The plot for $|H_2(\omega)|$ is a frequency-contracted version of the plot for $|H(\omega)|$: by a factor of 2.



$\Rightarrow$



MT1.5 (25 Points) The impulse response of a discrete-time LTI filter is described as follows:

$$\forall n \in \mathbb{Z}, \quad h(n) = \frac{\delta(n) - \delta(n-2)}{2}.$$

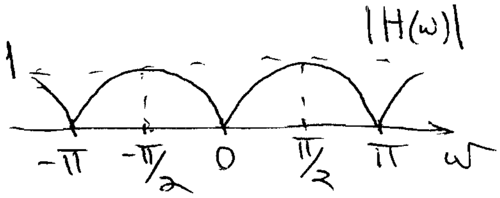
Determine a reasonably simple expression for the frequency response values $H(\omega)$, the magnitude response values $|H(\omega)|$, and the phase response values $\angle H(\omega)$.

Provide well-labeled plots for $|H(\omega)|$ and $\angle H(\omega)$ over the interval $-\pi < \omega < \pi$.

$$H(\omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-i\omega n} = \frac{1}{2}\left(1 - e^{-i2\omega}\right) = e^{-i\omega}\,\frac{e^{i\omega} - e^{-i\omega}}{2}$$

$$H(\omega) = i\sin(\omega)\, e^{-i\omega} = \sin(\omega) e^{i\left(\frac{\pi}{2} - \omega\right)} \qquad \left(\text{note: } i = e^{i\pi/2}\right)$$

$$|H(\omega)| = |\sin\omega|$$

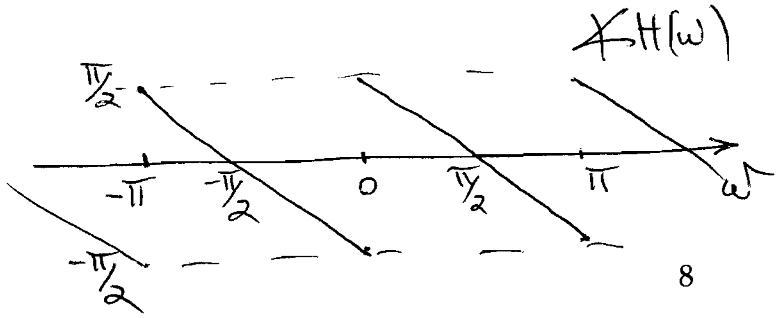


For the phase $\measuredangle H(\omega)$, we must consider the frequency intervals for which $\sin(\omega) > 0$ and $\sin(\omega) < 0$, separately.

$\sin(\omega) > 0$ for $0 < \omega < \pi$

$\sin(\omega) < 0$ " $-\pi < \omega < 0$

$$\measuredangle H(\omega) = \begin{cases} \frac{\pi}{2} - \omega & \text{if } 0 < \omega < \pi \quad (\text{where } \sin(\omega) > 0) \\ -\pi + \frac{\pi}{2} - \omega = \frac{-\pi}{2} - \omega & \text{if } -\pi < \omega < 0 \end{cases}$$



Note $\measuredangle H(\pm\frac{\pi}{2}) = 0$.
You should expect this.
Why?

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

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Lab Time Who needs lab?

Problem	Points	Your Score
Name	10	10
1	20	20
2	20	20
3	15	15
4	25	25
5	25	25
Total	**115**	115