EECS 20N: Structure and Interpretation of Signals and Systems   MIDTERM 2
Department of Electrical Engineering and Computer Sciences   02 November 2005
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name _Solutions_   FIRST Name _Sample_

Lab Time _Any day, any time!_

- (10 Points) Print your name and lab time in legible, block lettering in the appropriate spaces provided above AND on the back of the last page.

- This midterm should take you up to 80 minutes to complete. You will be given at least 80 minutes—up to a maximum of 90 minutes—to work on the midterm.

- Aside from TWO two-sided 8.5×11-inch formula sheet for your exclusive use, you may not access, or cause to be used or accessed, any reference in print or electronic form at any time during the quiz. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct.

- **The midterm printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the midterm is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.

- Please write neatly and legibly, because *if we can't read it, we can't grade it.*

- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in evaluating your midterm. No exceptions.*

- You must clearly and convincingly explain all your answers.

- We hope you wrestle and beat this midterm to the ground!

MT2.1 (20 Points) The impulse response g of a discrete-time, linear, time-invariant (LTI) system $G : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$ is given by:

$$g : \mathbb{Z} \to \mathbb{R}$$

$$\forall n \in \mathbb{Z}, \quad g(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}.$$

Determine the unit-step response s of the system G by plotting it and labeling all its salient features. That is, determine and plot the output of the system in response to the unit-step input signal u, where $\forall n \in \mathbb{Z}$,

$$u(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}.$$

DT-LTI

$$u \longrightarrow \boxed{g} \longrightarrow s$$

$\underline{n \geq 0}$

$$s(n) = (u \star g)(n) = \sum_{k=-\infty}^{\infty} u(k) g(n-k) = \sum_{k=0}^{n} 1 = n+1$$

$\leftarrow$ b/c $g(n-k) = 0 \quad \forall k > n$
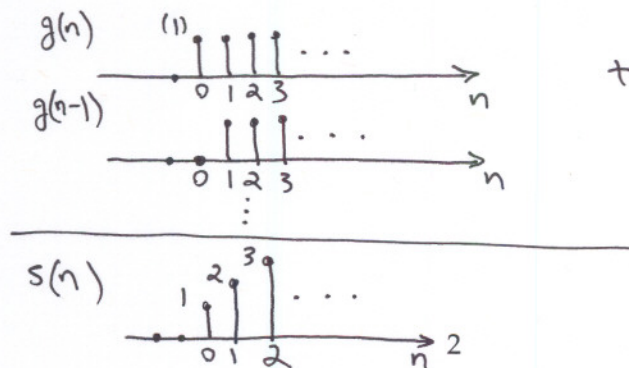
$\searrow$ b/c $u(k) = 0 \quad \forall k < 0$

$$s(n) = \begin{cases} n+1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$\underline{\text{Graphically:}}$ Using the "echo" view of convolution

$$s(n) = g(n) + g(n-1) + g(n-2) + \cdots$$
$$\overset{\text{or}}{=} u(n) + u(n-1) + u(n-2) + \cdots$$

$g(n)$

$g(n-1)$

$s(n)$

MT2.2 (20 points) A discrete-time system $F : [\mathbb{Z} \to \mathbb{R}] \to [\mathbb{Z} \to \mathbb{R}]$ is known to be *time invariant*. Three exemplary input-output behaviors $(x_i, y_i), i = 1, 2, 3$, of the system are depicted in the figure below (all signals are zero outside the regions shown):
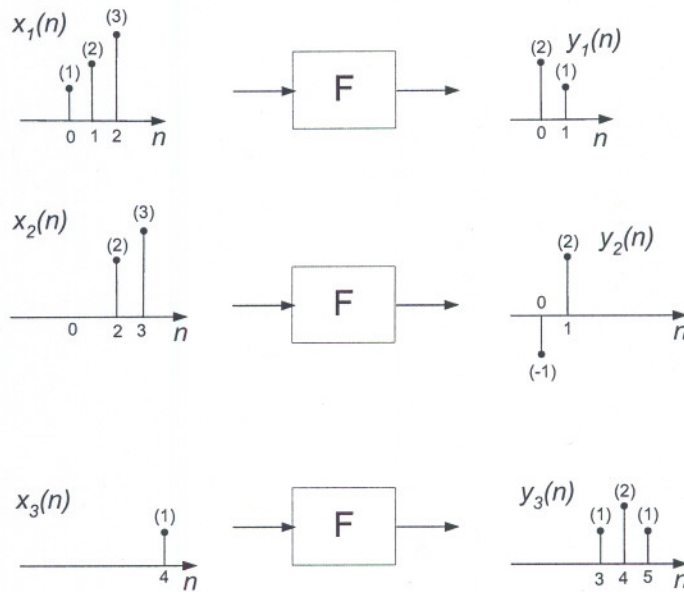
(a) Could the system F be causal? Explain your reasoning succinctly, but clearly and convincingly.

No! $\quad x_1(n) = x_2(n+1) \quad \forall n \leq 1$, but $y_2(n+1) \neq 0$ for $n = -1$

$\qquad\qquad\qquad\qquad\qquad$ (where as $y_1(n) = 0$ @ $n = -1$)

(b) Could the system F be memoryless? Explain your reasoning succinctly, but clearly and convincingly.

No! A system that is not causal cannot be memoryless.

Recall that: $\qquad$ Memoryless $\Longrightarrow$ Causal, which is logically

equivalent to: $\qquad$ Not Causal $\Longrightarrow$ Not Memoryless.

(c) Determine the output of the system F in response to the unit-impulse signal δ (Kronecker delta function), where

$$\delta : \mathbb{Z} \to \mathbb{R}$$

$$\forall n \in \mathbb{Z}, \qquad \delta(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}.$$

F is time invariant. Hence, knowing that $\delta(n) = x_3(n+4)$ means that the response of the system to the unit impulse is $(F(\delta))(n) = y_3(n+4)$
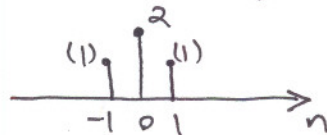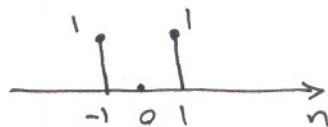

(d) Could the system F be linear? Explain your reasoning succinctly, but clearly and convincingly.

Not linear! $\qquad \delta(n) = x_1(n) - x_2(n+1)$

F is TI $\Longrightarrow$ F's response to $\hat{x}_2$, s.t. $\hat{x}_2(n) = x_2(n+1)$ is $\hat{y}_2$, s.t. $\hat{y}_2(n) = y_2(n+1)$.

Assume F is linear. Then $y_1(n) - y_2(n+1)$ is plotted below:

This is not the same as $(F(\delta))(n)$ that we found in part (c). We have a contradiction. Therefore, F can not be linear.

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MT2.3 (20 points) Consider a causal, discrete-time, SISO LTI system whose $[A, B, C, D]$ state-space representation includes the following state-update equation:

$$\underbrace{\begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix}}_{s(n+1)} = \underbrace{\begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix}}_{s(n)} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} x(n) .$$

The input signal and the state response are $x : \mathbb{N}_0 \to \mathbb{R}$ and $s : \mathbb{N}_0 \to \mathbb{R}$, respectively. Throughout this problem, let the input signal x be zero (i.e., $x(n) = 0$, $\forall n \geq 0$), and consider the initial state given by ($\exists \alpha, \beta \in \mathbb{R}$):

$$s(0) = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 3 \end{bmatrix} .$$

(a) Determine a simple expression for $s(n)$, the state of the system at time n ($\forall n \geq 1$).

$$s(1) = A\,s(0) = \alpha \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} \frac{1}{2} & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \alpha \left(\tfrac{1}{2}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\,(2)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\vdots$$

$$s(n) = \alpha \left(\tfrac{1}{2}\right)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\,(2)^n \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(b) Let $\alpha = 2$ and $\beta = 1$. Is the state $\begin{bmatrix} 5 & 5 \end{bmatrix}^T$ reachable at any time n? If so, at what time instant n can it be reached? Explain your reasoning succinctly, but clearly and convincingly.

$$s(n) = 2\left(\tfrac{1}{2}\right)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2^n \begin{bmatrix} 2 \\ 3 \end{bmatrix} \implies s(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$\forall n > 1$, the second component of $s(n)$ will be larger than the second component of $s(1)$ (which is 6). Therefore, $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$ is not reachable.

(c) Determine all values of α and β such that the zero-input state response decays to zero as $n \to \infty$. Please note that the set of real numbers $\mathbb{R}$ does not include $\pm\infty$.

$s(n)$ has two terms: a "stable" term $\alpha \left(\tfrac{1}{2}\right)^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which decays to zero as $n \to \infty$; and an "unstable" term $\beta\,(2)^n \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, which grows exponentially as $n \to \infty$. Hence, we must have:

$$\alpha \in \mathbb{R}, \quad \beta = 0$$

MT2.4 (20 points) A real, causal, discrete-time linear time-invariant (LTI) system is characterized by the following delay-adder-gain block diagram. The input and output at time n are denoted by $x(n)$ and $y(n)$, respectively. Each block D corresponds to a delay by one sample; that is, if the input to the delay block D is a signal r, the output of the delay block is the signal q, where $q(n) = r(n-1)$ for all n.

$$s_2(n+1) \qquad\qquad s_1(n+1)$$
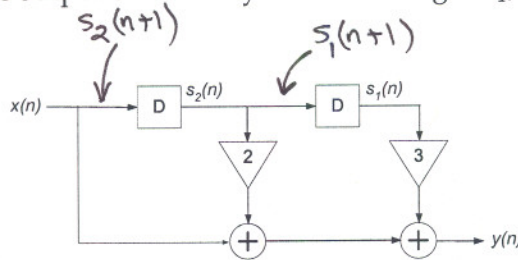
(a) Determine the linear, constant-coefficient difference equation that governs the input-output behavior of the system.

$$y(n) = x(n) + 2x(n-1) + 3x(n-2)$$

(b) Determine and provide a well-labeled plot of the impulse response h of the system.

$$h(n) = \delta(n) + 2\delta(n-1) + 3\delta(n-2)$$

(c) The outputs of the delay blocks are selected as the state variables $s_1(n)$ and $s_2(n)$, as shown in the figure. For this selection of state variables, determine the corresponding $[A, B, C, D]$ state-space representation of the LTI system.

$$s_1(n+1) = s_2(n)$$
$$s_2(n+1) = x(n)$$

$$\implies \begin{bmatrix} s_1(n+1) \\ s_2(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} x(n)$$

$$y(n) = x(n) + 2s_2(n) + 3s_1(n) \implies y(n) = \underbrace{\begin{bmatrix} 3 & 2 \end{bmatrix}}_{C} \begin{bmatrix} s_1(n) \\ s_2(n) \end{bmatrix} + \underset{\underset{D}{\uparrow}}{1} \cdot x(n)$$

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MT2.5 (20 points) Consider a discrete-time signal $x : \mathbb{Z} \to \mathbb{R}$ having the following the properties:

- $x(n + 4l) = x(n), \; \forall l \in \mathbb{Z}$.

- $\sum_{n=-1}^{2} x(n) = 2$.

- $\sum_{n=-1}^{2} (-1)^n x(n) = 4$.

- $\sum_{n=-1}^{2} x(n) \cos(\frac{\pi}{2}n) = \sum_{n=-1}^{2} x(n) \sin(\frac{\pi}{2}n) = 0$.

Determine the complex exponential Fourier series coefficients $X_{-1}, X_0, X_1,$ and X_2 for the signal x. From the coefficients, determine and provide a well-labeled plot for the signal x.

The following complex exponential Fourier series expressions for a periodic discrete-time signal having period p may be of potential use to you:

$$x(n) = \sum_{k=\langle p \rangle} X_k e^{ik\omega_0 n} \qquad X_k = \frac{1}{p} \sum_{n=\langle p \rangle} x(n) e^{-ik\omega_0 n} ,$$

where $\omega_0 = \dfrac{2\pi}{p}$ and $\langle p \rangle$ denotes a suitable contiguous discrete interval of length p.

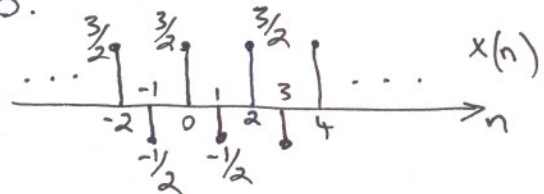$$P = 4 \implies \omega_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$X_0 = \frac{1}{4} \sum_{n=-1}^{2} x(n) = \frac{1}{4}(2) = \frac{1}{2}$$

$$X_2 = \frac{1}{4} \sum_{n=-1}^{2} x(n) e^{-i2\frac{\pi}{2}n} = \frac{1}{4} \sum_{n=-1}^{2} (-1)^n x(n) = \frac{1}{4}(4) = 1$$

$$X_1 = \frac{1}{4} \sum_{n=-1}^{2} x(n) e^{-i\frac{\pi}{2}n} = \frac{1}{4}\left(\underbrace{\sum_{n=-1}^{2} x(n) \cos(\frac{\pi}{2}n)}_{0} - i \underbrace{\sum_{n=-1}^{2} x(n) \sin(\frac{\pi}{2}n)}_{0} \right) = 0$$

$X_{-1} = 0$ for same reason as $X_1 = 0$.

Hence, $x(n) = \frac{1}{2} + (-1)^n$

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MT2.6 (20 points) This problem explores some of the properties of finite-state machines. Throughout, let the set $D = \{0, 1, \text{absent}\}$ denote an alphabet. Then each relevant input alphabet will be in D^2. For every input pair $(x_1(n), x_2(n)) \in D^2$, $x_1(n)$ and $x_2(n)$ denote the top and bottom input symbols, respectively.
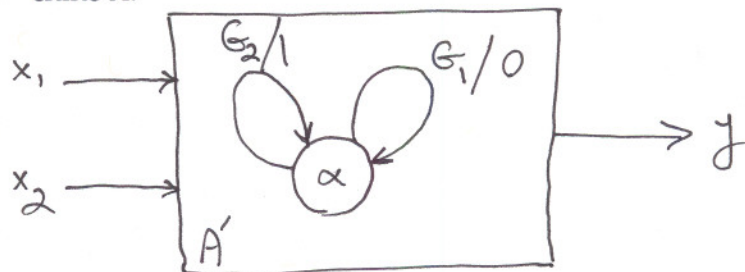
(a) Consider the finite-state machine A shown below.
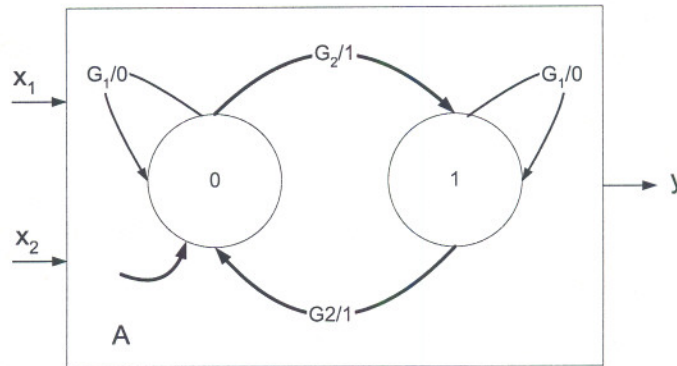
(i) Is machine A memoryless? Explain your reasoning succinctly, but clearly and convincingly.
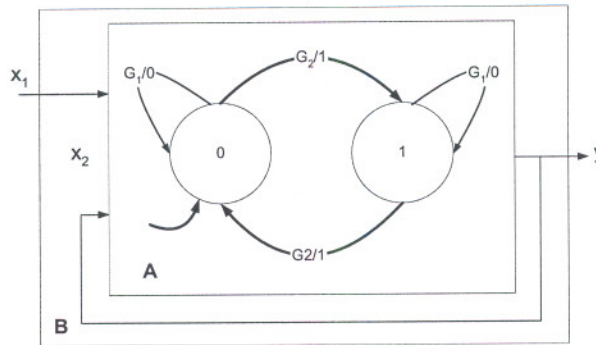
Yes. It doesn't matter what state you're in. This machine is bisimilar to a one-state machine (see part (ii)).

(ii) Find a simpler deterministic state machine that is bisimilar to machine A.

The answer to this part should have confirmed your answer to part (i).

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(b) Finite-state machine B is constructed by placing machine A in the feedback composition shown below:

(i) Which of the following guard sets G_1, G_2 makes machine B well-formed?

$$(I) \begin{cases} G_1 = \{(1,0)\} \\ G_2 = \{(1,1)\} \end{cases} \qquad (II) \begin{cases} G_1 = \{\} \\ G_2 = \{(0,1),(1,1)\} \end{cases}$$

$$(III) \begin{cases} G_1 = \{(1,0),(0,0)\} \\ G_2 = \{\} \end{cases} \qquad (IV) \begin{cases} G_1 = \{(1,0),(1,1),(0,1)\} \\ G_2 = \{(0,0)\} \end{cases}$$

(I) B is not well-formed because for the same $x_1 = 1$, there are two non-stuttering fixed points.

(II), (III): B is well-formed because for each nonstuttering input x_1, there is a unique fixed pt: (II) $x_1 = 0 \rightarrow y = x_2 = 1$
$x_1 = 1 \rightarrow y = x_2 = 1$

(III): $x_1 = 0 \rightarrow y = x_2 = 0$; $x_1 = 1 \rightarrow y = x_2 = 0$.

(IV) B is NWF. No nonstuttering fixed point for $x_1 = 0$.

(ii) Assuming G_1 and G_2 are such that machine B is well-formed, can you conclude that B is memoryless? Explain your reasoning succinctly, but clearly and convincingly.

If B is well-formed, it is bisimilar to one of the following deterministic single-state machines:
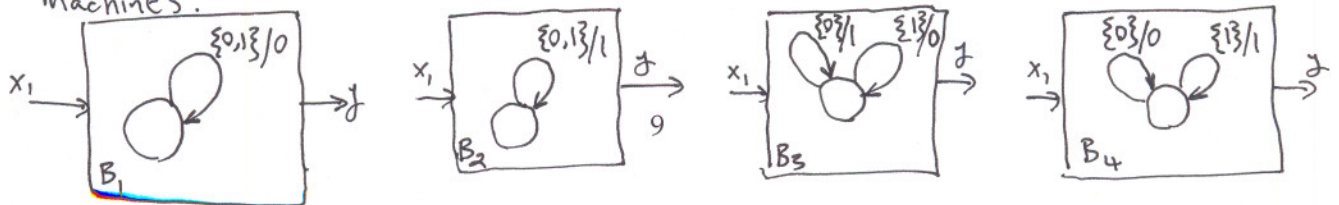
Each of these is memoryless.

LAST Name _Solutions_ FIRST Name _Sample_
Lab Time _Any day, any time_

Problem	Points	Your Score
Name	10	10
1	20	20
2	20	20
3	20	20
4	20	20
5	20	20
6	20	20
Total	**130**	130