ME 185/287 Midterm #2

1. **(6 pts)** Recall Reynolds' transport theorem

$$\frac{d}{dt}\int_{P_t} \phi dv = \int_{P_t} \phi' dv + \int_{\partial P_t} \phi \mathbf{v}\cdot\mathbf{n} da,$$

where the time derivative on the left-hand side is carried out for a fixed set S of material points, the notation $()'$ stands for the time derivative at fixed spatial position $\mathbf{x}$, and $\mathbf{v}(\mathbf{x},t)$ is the material velocity field.

(a) Using Reynolds' transport theorem, show that the conservations of mass and linear momentum, namely

$$\frac{d}{dt}\int_{P_t} \rho dv = 0$$

and

$$\frac{d}{dt}\int_{P_t} \rho v_i dv = \int_{\partial P_t} T_{ij} n_j da + \int_{P_t} \rho b_i dv,$$

may be cast in the forms

$$\int_{P_t} [\rho' + div(\rho\mathbf{v})] dv = 0$$

and

$$\int_{P_t} (\rho v_i)' dv = \int_{P_t} [(T_{ij} - \rho v_i v_j)_{,j} + \rho b_i] dv,$$

respectively. Here the subscripts refer to components of vectors and tensors with respect to a fixed orthonormal basis, and the notation $(\cdot)_{,j}$ stands for $\partial(\cdot)/\partial x_j$, where x_i $(i = 1, 2, 3)$ are the corresponding Cartesian coordinates of $\mathbf{x}$.

(b) Use these results together with the Localization Theorem to obtain the pointwise balance equations

$$\rho' + div(\rho\mathbf{v}) = 0 \quad \text{and} \quad (\rho\mathbf{v})' = div(\mathbf{T} - \rho\mathbf{v}\otimes\mathbf{v}) + \rho\mathbf{b}$$

holding at all points in the interior of κ_t.

(c) **(Extra credit)** Obtain the result of part (b) directly from the mass-conservation equation and the local equation of motion

$$div\mathbf{T} + \rho\mathbf{b} = \rho\dot{\mathbf{v}}.$$

2. **(7 pts)** Consider the velocity field given, in terms of polar coordinates and in the spatial description, by

$$\mathbf{v} = r\omega\mathbf{e}_\theta(\theta) + w(r)\mathbf{k},$$

where ω is a constant and $w(r)$ is a given function. Note that this velocity field is *steady*, i.e., $\mathbf{v}' = \mathbf{0}$.

(a) Derive the expression

$$\mathbf{L} = [\omega\mathbf{e}_\theta + w'(r)\mathbf{k}]\otimes\mathbf{e}_r - \omega\mathbf{e}_r\otimes\mathbf{e}_\theta$$

for the spatial velocity gradient.

(b) Use $\dot{J} = J tr\mathbf{L}$ to show that $\dot{J} = 0$ and hence that the spatial form of the mass conservation equation reduces to

$$\rho' + \mathbf{v} \cdot \nabla\rho = 0,$$

where ρ' and $\nabla\rho$ respectively are the time derivative and gradient in the spatial description.

(c) Consider a material curve with unit tangent $\mathbf{m}$ in κ_t. Show that the stretch μ of this material curve has the material derivative $\dot{\mu} = 0$ in each of the three cases $\mathbf{m} = \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{k}$.

3. **(7 pts)** The Cauchy stress in an inviscid compressible gas is of the form

$$\mathbf{T} = -p(\rho)\mathbf{I},$$

where $p(\rho)$ is the pressure-density relation. For example, in an ideal gas this relation is $p(\rho) = k\rho$, where k is a constant that characterizes the particular ideal gas considered.

(a) Derive the relation

$$div\mathbf{T} = -p'(\rho)\nabla\rho,$$

where $p'(\rho) = dp/d\rho$.

(b) Assuming zero body force, show that the equation of motion for a gas undergoing the motion of Problem # 2 reduces to

$$p'(\rho)\nabla\rho = \rho r\omega^2\mathbf{e}_r.$$

Hence, assuming that $p'(\rho) \neq 0$, show that $\rho(\mathbf{x}, t)$ is steady, i.e., its spatial time derivative is $\rho' = 0$. Further, show that ρ can depend only on the radius r.

(c) Assume an ideal gas and derive an expression for the function $\rho(r)$. Impose the condition $\rho(0) = \rho_0$, a given constant. [Note: the density is not affected by the axial component $w(r)$ of the velocity field.]