ME 185/287 Midterm 2 Solutions

1. (a) Let $\phi = \rho$. Then,

$$0 = \tfrac{d}{dt}\int_{P_t} \rho dv = \int_{P_t} \rho' dv + \int_{\partial P_t} \rho \mathbf{v}\cdot\mathbf{n} da = \int_{P_t} [\rho' + div(\rho\mathbf{v})] dv.$$

Next, let $\phi = \rho v_i$. Then,

$$\begin{aligned}\int_{P_t} [(\rho v_i)' + (\rho v_i v_j)_{,j}] dv &= \int_{P_t} (\rho v_i)' dv + \int_{\partial P_t} \rho v_i v_j n_j da \\ &= \tfrac{d}{dt}\int_{P_t} \rho v_i dv \\ &= \int_{\partial P_t} T_{ij} n_j da + \int_{P_t} \rho b_i dv \\ &= \int_{P_t} (T_{ij,j} + \rho b_i) dv,\end{aligned}$$

which is the stated result.

(b) As $P_t \subset \kappa_t$ is arbitrary, we can localize to obtain

$$\rho' + div(\rho\mathbf{v}) = 0 \quad \text{and} \quad (\rho v_i)' = (T_{ij} - \rho v_i v_j)_{,j} + \rho b_i,$$

the second of which is the $\mathbf{e}_i$ - component of the stated result. These hold at all $\mathbf{x} \in \kappa_t$.

(c) We have

$$\begin{aligned}T_{ij,j} + \rho b_i &= \rho \dot{v}_i = \rho v_i' + \rho v_{i,j} v_j \\ &= (\rho v_i)' - \rho' v_i + \rho v_{i,j} v_j \\ &= (\rho v_i)' + (\rho v_j)_{,j} v_i + \rho v_{i,j} v_j \\ &= (\rho v_i)' + (\rho v_i v_j)_{,j}.\end{aligned}$$

2. (a) We have $\mathbf{v} = r\omega\mathbf{e}_\theta + w'(r)\mathbf{k}$, and therefore

$$\mathbf{L}d\mathbf{x} = d\mathbf{v} = \omega dr \mathbf{e}_\theta + r\omega d\mathbf{e}_\theta + w'(r) dr \mathbf{k} = \omega\mathbf{e}_\theta(\mathbf{e}_r \cdot d\mathbf{x}) - \omega\mathbf{e}_r(\mathbf{e}_\theta \cdot d\mathbf{x}) + w'(r)\mathbf{k}(\mathbf{e}_r \cdot d\mathbf{x}),$$

so

$$\mathbf{L} = \omega(\mathbf{e}_\theta \otimes \mathbf{e}_r - \mathbf{e}_r \otimes \mathbf{e}_\theta) + w'(r)\mathbf{k} \otimes \mathbf{e}_r,$$

as claimed.

(b) Note that $\dot{J}/J = tr\mathbf{L} = 0$. Thus, $\dot{J} = 0$ and mass conservation yields $0 = \dot{\rho}_\kappa = (\rho J)^{\cdot} = \dot{\rho} J$; then,

$$0 = \dot{\rho} = \rho' + \nabla\rho \cdot \mathbf{v}.$$

(c) We have $\mu\mathbf{m} = \mathbf{F}\mathbf{M}$ and therefore $\dot{\mu}\mathbf{m} + \mu\dot{\mathbf{m}} = \dot{\mathbf{F}}\mathbf{M} = \mu\dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{m} = \mu\mathbf{L}\mathbf{m}$. Then, $\dot{\mu} = \mu\mathbf{m}\cdot\mathbf{L}\mathbf{m}$ (since $\mathbf{m}\cdot\mathbf{m} = 1$; hence, $\mathbf{m}\cdot\dot{\mathbf{m}} = 0$). For $\mathbf{m} = \mathbf{e}_r, \mathbf{e}_\theta$ or $\mathbf{k}$ the above expression for $\mathbf{L}$ gives $\dot{\mu} = 0$.

3. We have

$$\mathbf{T} = -p(\rho)\mathbf{I}, \quad \text{or} \quad T_{ij} = -p\delta_{ij},$$

in terms of components.

(a) Then,

$$T_{ij,j} = -(p\delta_{ij})_{,j} = -p_{,j}\delta_{ij} = -p_{,i} = -p'(\rho)\rho_{,i},$$

which is the Cartesian component form of

$$div\mathbf{T} = -p'(\rho)\nabla\rho.$$

(b) We need to solve

$$div\mathbf{T} = \rho\dot{\mathbf{v}} = \rho(\mathbf{v}' + \mathbf{L}\mathbf{v}).$$

For the motion of Problem # 2 we have $\mathbf{v}' = \mathbf{0}$ and $\mathbf{L}\mathbf{v} = -r\omega^2\mathbf{e}_r$. Thus,

$$p'(\rho)\nabla\rho = \rho r\omega^2\mathbf{e}_r.$$

Assuming $p'(\rho) \neq 0$, this means that $\mathbf{v}\cdot\nabla\rho = 0$, and the result of Problem # 2(b) yields $\rho' = 0$. Further, using

$$\nabla\rho = (\partial\rho/\partial r)\mathbf{e}_r + r^{-1}(\partial\rho/\partial\theta)\mathbf{e}_\theta + (\partial\rho/\partial z)\mathbf{k}$$

we obtain $\partial\rho/\partial\theta = 0 = \partial\rho/\partial z$ and $\partial\rho/\partial r = \rho'(r)$, where

$$p'(\rho)\rho'(r) = \rho r\omega^2.$$

(c) For an ideal gas we have $p'(\rho) = k$, a given constant. Thus,

$$(\ln\rho)' = \rho'(r)/\rho = r\omega^2/k.$$

Integrating and using $\rho(0) = \rho_0$ gives $\ln(\rho/\rho_0) = r^2\omega^2/2k$. Finally,

$$\rho(r) = \rho_0\exp(\tfrac{r^2\omega^2}{2k}).$$