Math 55 midterm exam, March 20, 2024

This exam consisted of five problems, each worth 7 points. The maximum possible score on the exam was thus 35.

A comment on the cover sheet: **It is OK to leave factorials and expressions like $C(n, r)$ or $P(n, r)$ in your answers.**

1. At least 500 of 600 graduating seniors came to a commencement. When these students were divided into groups of 6, 8 and 11, three students were left over in each case. How many students came to the commencement?
[For all problems, explain your reasoning in complete English sentences.]

Let n be the number of students who showed up. We are given that $n - 3$ is divisible by 6, 8 and 11. Thus $n-3$ is divisible by 3, 2^3 and 11 and so is divisible by $3 \cdot 2^3 \cdot 11 = 264$. (For this, we might use prime factorization.) The multiples of 264 are 528, 792, ... Because n is between 500 and 600, the only possibility for $n - 3$ is 528. Therefore $n = 531$.

2. Let p be a prime number other than 2, 3 or 5. Show that p divides the sum

$$10^{p-2} + 10^{p-3} + \cdots 10^2 + 10 + 1;$$

for example, 13 divides the 12-digit number 111111111111.
[It might be helpful to use the identity $x^n - 1 = (x-1)(1+x+x^2+\cdots+x^{n-1})$.]

By the identity in the hint,

$$10^{p-2} + 10^{p-3} + \cdots 10^2 + 10 + 1 = \frac{10^{p-1} - 1}{10 - 1}.$$

By Fermat's little theorem, the numerator of this fraction is divisible by p. Since $p > 3$, the denominator of the fraction (which is 9) is not divisible by p. Let $f = 10^{p-2} + 10^{p-3} + \cdots 10^2 + 10 + 1$ be the fraction, and write $f = n/d$, where $n = 10^{p-1} - 1$ is the numerator and $d = 9$ is the denominator. Then p divides $n = fd$. Because p does not divide d, Euclid's lemma implies that p divides f.

3. Consider 3-digit strings of distinct letters (e.g., `BER`, `KLY`, `STN`, `FRD`, `SFO`). How many such strings contain at least one vowel (`A`, `E`, `I`, `O`, `U`)?

There are 26 letters in the alphabet. Thus the number of 3-digit strings with distinct letters is $P(26, 3) = 26 \cdot 25 \cdot 24$. There are 5 vowels and thus 21 non-vowels. The number of 3-digit strings with distinct letters and no vowels is therefore $P(21,3) = 21 \cdot 20 \cdot 19$. The answer to the question in the problem is the difference $P(26,3) - P(21,3)$.

4. The inequality

$$x + y + z + w \leq 55 \qquad (*)$$

states that the difference $55 - (x + y + z + w)$ is nonnegative. Find the number of solutions to $(*)$ with x, y, z and w nonnegative integers.

Let $v = 55 - (x + y + z + w)$. Then the number of solutions to the inequality of the problem is the number of solutions to the *equality*

$$x + y + z + w + v = 55.$$

A bagel (= stars & bars) analysis shows that the number is $C(55 + 4, 4)$.

During the exam, it became clear that students were calculating the number of solutions to $x + y + z + w = k$ for each k between 0 and 55, and then summing up the results. This will yield a sum of binomial coefficients as the answewr. That's not what we had in mind, but it's OK. The fact that this sum is equal to $C(55+4, 4)$ is a special case of the hockey stick identity. If you do the problem in two ways (as done above and then also by writing the answer as a sum), you've proved the identity. The general case of the hockey stick identity is proved the same way — you just have to replace 55 by N (or whatever) and have another notation for the number of variables in the inequality.

5. Computing the gcd of 21 and 8 creates these equations:

$$\begin{aligned} 21 &= 2 \cdot 8 + 5, \\ 8 &= 1 \cdot 5 + 3, \\ 5 &= 1 \cdot 3 + 2, \\ 3 &= 1 \cdot 2 + 1, \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

Let a and b be positive integers for which the computation of $\gcd(a, b)$ also produces five equations. Explain in detail why b is at least 8.

We begin the Euclidean algorithm by dividing a by b. If we have five equations, they look in our usual notation like this:

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2,\\
r_1 &= r_2 q_2 + r_3,\\
r_2 &= r_3 q_3 + r_4,\\
r_3 &= r_4 q_4 + r_5,\\
r_4 &= r_5 q_5 + 0.
\end{aligned}
$$

Here, $a = r_0$, $b = r_1$, and then $r_1 > r_2 > r_3 > r_4 > r_5 > 0$ because the remainder in a division is less than the divisor in that division. (If r_5 were 0, then we would stop dividing and say that there are only four equations—not five.) Hence r_5 is at least 1 and r_4 is at least 2. The quotients q_i are all positive integers: if q_3 (for example) were 0, then r_2 would be equal to r_4. Hence $r_3 = r_4 q_4 + r_5$ is at least $2 \cdot 1 + 1 = 3$. Similarly, $r_2 = r_3 q_3 + r_4$ is at least $3 \cdot 1 + 2 = 5$. Finally, $r_1 = r_2 q_2 + r_3$ is at least $5 \cdot 1 + 3 = 8$.