EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 15 February 2007
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Philter FIRST Name Antinotch
Lab Time Midnight – 3AM

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (35 Points) You can tackle the two parts of this problem independently. Explain your responses succinctly, but clearly and convincingly.

(a) (20 Points) Determine whether the following argument is valid.

(I) If a system is linear *and* time invariant, then the output of the system can NOT precede the corresponding input. You will learn later that (I) is false.

(II) Every memoryless system is causal.

(III) The output of a *linear* system F precedes its corresponding input.

(IV) Therefore, system F is NOT memoryless.

Hint: You do not need to know—nor may you invoke—any knowledge of system properties to evaluate this argument. Instead, use a symbol to denote each assertion (e.g., L = The system is linear.) and apply principles of logic.

Let $L \triangleq$ system is linear ; $TI \triangleq$ System is Time Invariant;
$C \triangleq$ system is causal ; $M \triangleq$ System is Memoryless
$P \triangleq$ output precedes the input.

(I) $L \wedge TI \Rightarrow \neg P$ (logical equiv of $P \Rightarrow \neg(L \wedge TI) = (\neg L) \vee (\neg TI)$)

(II) $M \Rightarrow C$ $(\neg C \Rightarrow \neg M)$

(III) $L \wedge P \Rightarrow \neg L$ is false
$\Rightarrow P$ is true $\Rightarrow (\neg L) \vee (\neg TI)$ $\Rightarrow \neg TI$ must be true $\Rightarrow$ System F is not time invariant.

(IV) doesn't follow $\Rightarrow$ Argument invalid.

(b) (15 Points) Determine whether the following argument is valid.

(I) Every finite-energy signal has a well-defined Fourier transform.

(II) Signal $x : \mathbb{R} \to \mathbb{R}$ has infinite energy.

Therefore, signal x does *not* have a well-defined Fourier transform.

Hint: You do not need to know—nor may you invoke—any knowledge of signal energy or Fourier transforms to evaluate this argument.

Let $FE \triangleq$ signal has finite energy ; $FT \triangleq$ signal has a well-defined Fourier transform.

(I) $FE \Rightarrow FT$

(II) $\neg FE$

(III) $\neg FE \Rightarrow \neg FT$ this cannot be concluded from (I). In fact, negating the premise of (I) does not negate the consequent FT. This argument is not valid.

MT1.2 (40 Points) The frequency response $A : \mathbb{R} \to \mathbb{C}$ of a discrete-time filter A is characterized as follows:

$$\forall \omega \in \mathbb{R}, \quad A(\omega) = \frac{\alpha + e^{-i\omega}}{1 + \alpha\, e^{-i\omega}},$$

where $0 < \alpha < 1$ (note α is a real parameter).

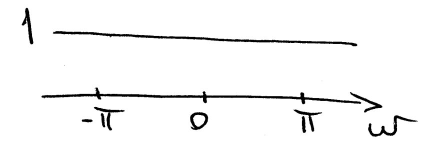
You may assume that every filter in this problem is a linear discrete-time system.

(a) (10 Points) Determine the magnitude response $|A(\omega)|$, $-\pi < \omega \leq +\pi$. Provide a well-labeled sketch of $|A(\omega)|$.

$$A(\omega) = \frac{\alpha + e^{-i\omega}}{1+\alpha e^{-i\omega}} = e^{-i\omega}\,\frac{1+\alpha e^{i\omega}}{1+\alpha e^{-i\omega}} \Rightarrow |A(\omega)| = \underbrace{\cancel{|e^{-i\omega}|}}_{1}\,\frac{|1+\alpha e^{i\omega}|}{|1+\alpha e^{-i\omega}|} \quad \Big\} \Rightarrow$$

Note that $1+\alpha e^{i\omega} = \left(1+\alpha e^{-i\omega}\right)^*$ for $\alpha \in \mathbb{R}$ $\Rightarrow |1+\alpha e^{i\omega}| = |1+\alpha e^{-i\omega}|$

$$\boxed{|A(\omega)| = 1} \quad \forall \omega$$



(b) (15 Points) The frequency response $G : \mathbb{R} \to \mathbb{C}$ of a discrete-time filter G is characterized as follows:

$$\forall \omega \in \mathbb{R}, \quad G(\omega) = \frac{1}{2}\left[1 + A(\omega)\right].$$

(i) Determine a reasonably simple expression for $G(\omega)$, $-\pi < \omega \leq +\pi$.

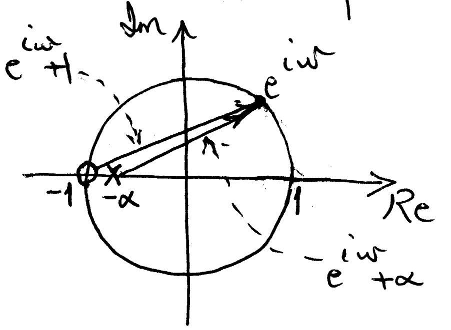
$$G(\omega) = \frac{1}{2}\left(1 + \frac{\alpha + e^{-i\omega}}{1+\alpha e^{-i\omega}}\right) = \frac{1}{2}\,\frac{1+\alpha e^{-i\omega} + \alpha + e^{-i\omega}}{1+\alpha e^{-i\omega}} = \frac{1}{2}\,\frac{(1+\alpha) + (1+\alpha)e^{-i\omega}}{1+\alpha e^{-i\omega}}$$

$$\boxed{G(\omega) = \frac{1+\alpha}{2}\,\frac{1+e^{-i\omega}}{1+\alpha e^{-i\omega}}} \quad \text{or} \quad \boxed{G(\omega) = \frac{1+\alpha}{2}\,\frac{e^{i\omega}+1}{e^{i\omega}+\alpha}}$$

(ii) Assume $\alpha = 0.95$ and provide a well-labeled sketch of the magnitude response $|G(\omega)|$, $-\pi < \omega \le +\pi$.

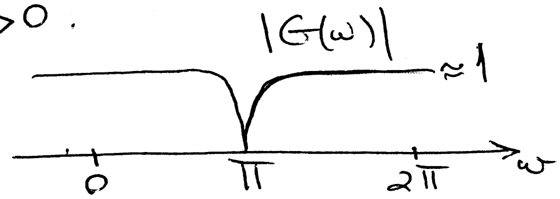
$\alpha = 0.95$, which essentially means $\frac{1+\alpha}{2}$ can be approximated to be 1. To see how the magnitude response behaves, look at the vectorial picture:

$$\underbrace{G(\omega) = \frac{1+\alpha}{2}\,\frac{e^{i\omega}+1}{e^{i\omega}+\alpha}}$$



The vectors representing $e^{i\omega}+1$ and $e^{i\omega}+\alpha$ are approximately equal except in the vicinity of -1, i.e., $e^{i\pi}$. As $\omega \to \pi$, $e^{i\omega}+1 \to 0$, whereas the denominator $e^{i\omega}+\alpha \to \alpha > 0$.

This is a notch filter



Extra: $\measuredangle G(\omega) \approx 1$ for all frequencies except those close to π.

(iii) Assume $\alpha = 0.95$ and determine a reasonable approximation for the output signal $y : \mathbb{Z} \to \cancel{\mathbb{R}}\ \mathbb{C}$ of the filter G, in response to the input signal $x : \mathbb{Z} \to \cancel{\mathbb{R}}\ \mathbb{C}$ where

$$\forall n \in \mathbb{Z}, \quad x(n) = 5 + (-1)^n + i^n + \cos\left(\frac{2\pi}{3}n\right).$$

The filter "notches out" any component @ around $\omega = \pi$. The only component of x at, or near, freqency π is $(-1)^n \triangleq e^{i\pi n}$. Everything else stays.

$$x(n) = 5 + (-1)^n + i^n + \cos\left(\tfrac{2\pi}{3}n\right) = \overset{\text{freq}=0}{5e^{i0n}} + \overset{\text{freq}=\pi}{e^{i\pi n}} + \overset{\text{freq}=\pi/2}{e^{i\frac{\pi}{2}n}} + \overset{\text{freq}=\frac{2\pi}{3}}{\tfrac{1}{2}e^{i\frac{2\pi}{3}n}} + \overset{\text{freq}=-\frac{2\pi}{3}}{\tfrac{1}{2}e^{-i\frac{2\pi}{3}n}}$$

$$y(n) \approx 5 + e^{i\frac{\pi}{2}n} + \frac{1}{2}e^{i\frac{2\pi}{3}n} + \frac{1}{2}e^{-i\frac{2\pi}{3}n} \implies$$

$$\underbrace{y(n) \approx 5 + i^n + \cos\left(\tfrac{2\pi}{3}n\right)}$$

(c) (15 Points) The frequency response $H : \mathbb{R} \to \mathbb{C}$ of a discrete-time filter H is characterized as follows:

$$\forall \omega \in \mathbb{R}, \quad H(\omega) = \frac{1}{2}\left[1 - A(\omega)\right].$$

Determine a mathematical relationship between the frequency responses G and H. Then use that relationship to provide a well-labeled sketch of the magnitude response $|H(\omega)|$, $-\pi < \omega \le +\pi$.

If you are unable to determine a relationship between G and H, then simply determine H based on A—in a manner similar to what you did for part (b)—and then provide a well-labeled sketch of $|H(\omega)|$.

Note $G(\omega) = \frac{1}{2}\left[1 + A(\omega)\right]$

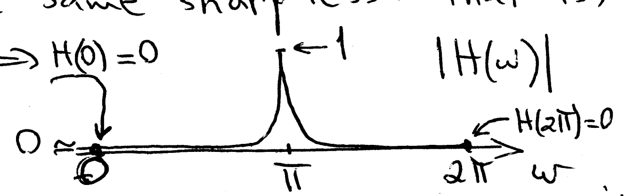
$H(\omega) = \frac{1}{2}\left[1 - A(\omega)\right]$

$$G(\omega) + H(\omega) = 1 \implies H(\omega) = 1 - G(\omega)$$

For $\forall \omega$ not near π, we know $G(\omega) \approx 1 \implies H(\omega) \approx 0$

Where $G(\omega)$ approaches zero as $\omega \to \pi$, $H(\omega)$ peaks to 1 with the same sharpness. That is,

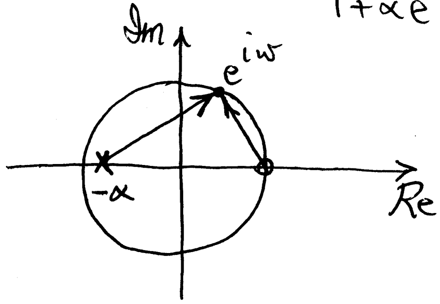
$G(0) = 1 \implies H(0) = 0$



This is called an Anti-Notch Filter

$$H(\omega) = \frac{1}{2}\left[1 - A(\omega)\right] = \frac{1}{2}\left(1 - \frac{\alpha + e^{-i\omega}}{1 + \alpha e^{-i\omega}}\right) = \frac{1}{2}\,\frac{1 + \alpha e^{-i\omega} - \alpha - e^{-i\omega}}{1 + \alpha e^{-i\omega}} = \frac{1}{2}\,\frac{(1-\alpha) - (1-\alpha)e^{-i\omega}}{1 + \alpha e^{-i\omega}}$$

$$H(\omega) = \frac{1-\alpha}{2}\,\frac{1 - e^{-i\omega}}{1 + \alpha e^{-i\omega}} \Rightarrow H(\omega) = \frac{1-\alpha}{2}\,\frac{e^{i\omega} - 1}{e^{i\omega} + \alpha}$$



$\frac{1-\alpha}{2} \approx 0 \Rightarrow$ For most ω (those not in the vicinity of π), $H(\omega) \approx \frac{1-\alpha}{2} \approx 0$.

Near $\omega = \pi$, the denominator term $e^{i\omega} + \alpha$ is approximately on the same order as $1 - \alpha$.

MT1.3 (20 Points) Consider each of the signal spaces $\mathsf{X}_{\mathsf{I}}, \dots, \mathsf{X}_{\mathsf{IV}}$ described below.

$$\mathsf{X}_{\mathsf{I}} = [\mathbb{R} \to \mathbb{R}] \qquad \mathsf{X}_{\mathsf{II}} = [\mathbb{R} \to \mathbb{R}]^2$$

$$\mathsf{X}_{\mathsf{III}} = [\mathbb{R}^2 \to \mathbb{R}] \qquad \mathsf{X}_{\mathsf{IV}} = [\mathbb{R}^2 \to \mathbb{R}]^2$$

Each of the following signals $f_A, \dots, f_D$ may belong to *at most* one of the signal spaces $\mathsf{X}_{\mathsf{I}}, \dots, \mathsf{X}_{\mathsf{IV}}$ above.

Match each function with a signal space, or explain why no such match exists. Justify each of your choices succinctly, but clearly and convincingly.

(a)

$$f_A(t) = \begin{bmatrix} \cos(120\pi t) \\ \sin(120\pi t) \end{bmatrix}, \quad \forall t \in \mathbb{R}.$$

$\cos(120\pi t) \in \mathbb{R}$, $\sin(120\pi t) \in \mathbb{R} \Rightarrow f_A(t) \in \mathbb{R}^2 \Rightarrow$

$f_A \in \underbrace{[\mathbb{R} \to \mathbb{R}]^2}_{X_{II}}$

(b)

$$f_B(x_1, x_2) = \cos(x_1^2 + x_2^2), \quad \forall x_1, x_2 \in \mathbb{R}.$$

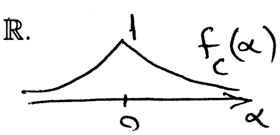
$\cos(x_1^2 + x_2^2) \in \mathbb{R} \Rightarrow f_B(x_1, x_2) \in \mathbb{R} \Rightarrow f_B \in \overbrace{[\mathbb{R}^2 \to \mathbb{R}]}^{X_{III}}$

$x_1, x_2 \in \mathbb{R}$

(c)

$$f_C(\alpha) = i^{i|\alpha|}, \quad \forall \alpha \in \mathbb{R}.$$

$f_C(\alpha) = i^{i|\alpha|} = \left(e^{i\pi/2}\right)^{i|\alpha|} = e^{-\frac{\pi}{2}|\alpha|} \in \mathbb{R}$



$\Rightarrow f_C \in [\mathbb{R} \to \mathbb{R}]$

(d)

$$f_D(x_1, x_2) = \begin{bmatrix} \cos(x_1^2 + x_2^2) \\ \sin(x_1^2 + x_2^2) \end{bmatrix}, \quad \forall x_1, x_2 \in \mathbb{R}.$$

Similar to f_A but the domain of the cos & sin in $\mathbb{R}^2$. Hence,

$f_D \in [\mathbb{R}^2 \to \mathbb{R}]^2$

MT1.4 (10 Points) You may tackle each of the two parts of this problem independently.

(a) True or false?

$$[\mathbb{R}^2 \to \mathbb{R}]^2 = [\mathbb{R}^2 \to \mathbb{R}^2].$$

Explain your reasoning succinctly, but clearly and convincingly.

The left-hand side:

$[\mathbb{R}^2 \to \mathbb{R}]^2 = [\mathbb{R}^2 \to \mathbb{R}] \times [\mathbb{R}^2 \to \mathbb{R}] = \{(f_1, f_2) \mid f_1 \in [\mathbb{R}^2 \to \mathbb{R}] \wedge f_2 \in [\mathbb{R}^2 \to \mathbb{R}]\}$

$= \{(f_1, f_2) \mid f_1 : \mathbb{R}^2 \to \mathbb{R} \wedge f_2 : \mathbb{R}^2 \to \mathbb{R}\}$

The righthand side: Let $f \in [\mathbb{R}^2 \to \mathbb{R}^2] \Rightarrow f : \mathbb{R}^2 \to \mathbb{R}^2$, $\forall x \in \mathbb{R}^2, f(x) \in \mathbb{R}^2 \Rightarrow$

$f(x) = (f_1(x), f_2(x))$, $\forall x \in \mathbb{R}$, where $f_1(x) \in \mathbb{R} \wedge f_2(x) \in \mathbb{R} \Rightarrow f = (f_1, f_2)$ where $f_1 : \mathbb{R}^2 \to \mathbb{R} \wedge f_2 : \mathbb{R}^2 \to \mathbb{R}$

Hence, $[\mathbb{R}^2 \to \mathbb{R}^2] = \{f \mid f = (f_1, f_2) \wedge f_1 : \mathbb{R}^2 \to \mathbb{R} \wedge f_2 : \mathbb{R}^2 \to \mathbb{R}\} = \{(f_1, f_2) \mid f_1 : \mathbb{R}^2 \to \mathbb{R} \wedge f_2 : \mathbb{R}^2 \to \mathbb{R}\}$

$= [\mathbb{R}^2 \to \mathbb{R}]^2$

(b) Consider a function $f : \mathsf{A} \to \mathsf{B}$, where $\mathsf{B} \subset \mathsf{A}$.

Define a function g as follows:

$$g = \underbrace{f \circ f \circ f \circ \cdots \circ f}_{N \text{ times, where } N \in \mathbb{N}}.$$

Which one of the following descriptions of g is correct?

(I) $g : \mathsf{A} \to \mathsf{B}$.

(II) $g : \mathsf{A}^N \to \mathsf{B}$.

(III) $g : \mathsf{A} \to \mathsf{B}^N$.

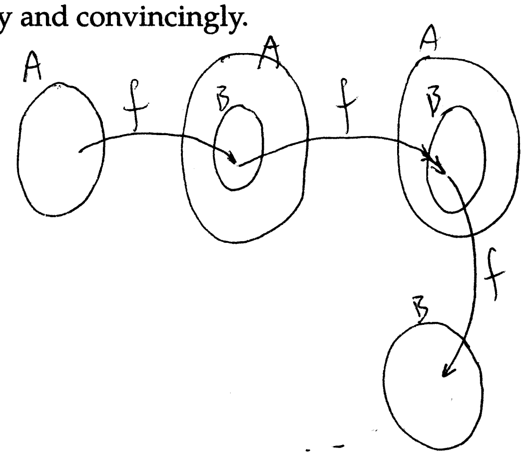
(IV) $g : \mathsf{A}^N \to \mathsf{B}^N$.

Explain your reasoning succinctly, but clearly and convincingly.

$f : A \to B$

$f \circ f : \underbrace{A \to B \to B}_{A \to B}$

Extends to arbitrary $N \in \mathbb{N}$.



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Problem	Points	Your Score
Name	10	10
1	35	35
2	40	40
3	20	20
4	10	10
Total	**115**	115