MATH 1B MIDTERM 2 (LEC 001)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e \qquad |R_N(x)| \leq \frac{M|x-a|^{N+1}}{(N+1)!}$$

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

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This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (30 points) Determine the convergence or divergence of the following infinite series:

(a)

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{e^{1/n}}$$

Solution:

$$\lim_{n\to\infty}\frac{1}{e^{1/n}} = \frac{1}{e^0} = 1 \neq 0 \Rightarrow \lim_{n\to\infty}(-1)^n\frac{1}{e^{1/n}} \neq 0$$

$$\underset{\text{D.T.}}{\Rightarrow} \underline{\text{divergent}}$$

(b)

$$\sum_{n=1}^{\infty}\tan\left(\frac{1}{n^2+1}\right)$$

Solution:

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2} \quad \text{for all } n = 1,2,3\ldots$$

p-series, $p=2>1$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \text{ conv} \underset{\text{S.C.T.}}{\Rightarrow} \sum_{n=1}^{\infty}\frac{1}{n^2+1} \ \underline{\text{conv}}$$

$$\lim_{n\to\infty}\frac{1}{n^2+1} = 0 \ \underline{\text{and}} \ \lim_{x\to 0}\frac{\tan(x)}{x} = \lim_{x\to 0}\frac{\sec^2(x)}{1} = 1$$

$$\Rightarrow \lim_{n\to\infty}\frac{\tan\left(\frac{1}{n^2+1}\right)}{\left(\frac{1}{n^2+1}\right)} = 1 > 0 \underset{\text{L.C.T.}}{\Rightarrow} \sum_{n=1}^{\infty}\tan\left(\frac{1}{n^2+1}\right) \ \underline{\text{conv}}$$

2. (30 points) The sequence $b_1, b_2, b_3, \dots$ has limit equal to 4. Determine the **radius of convergence** of the following power series.

$$\sum_{n=1}^{\infty} \underbrace{\frac{(2x+1)^{2n}}{9^n \cdot b_1 \cdot b_2 \cdots b_n}}_{a_n}$$

Solution:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{(2x+1)^{2n+2}}{9^{n+1} b_1 \cdots b_n b_{n+1}}\right)}{\left(\frac{(2x+1)^{2n}}{9^n b_1 \cdots b_n}\right)}\right| = \frac{1}{9} \cdot \frac{1}{b_{n+1}} \cdot |2x+1|^2$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{9} \cdot \frac{1}{4} \cdot |2x+1|^2 = \frac{|2x+1|^2}{36}$$

Ratio Test $\Rightarrow$
$$\begin{cases} \underline{\text{conv}} & \text{if } \frac{|2x+1|^2}{36} < 1 \\ \underline{\text{div}} & \text{if } \frac{|2x+1|^2}{36} > 1 \end{cases}$$

$$\frac{|2x+1|^2}{36} < 1 \iff |2x+1|^2 < 36 \iff |2x+1| < 6$$

$$\iff 2\left|x+\tfrac{1}{2}\right| < 6 \iff \left|x + \tfrac{1}{2}\right| < 3$$

$$\Rightarrow \quad \text{R.O.C.} = 3$$

3. (30 points) Using the **integral test**, and any other relevant tests, determine whether the following infinite series is absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{n^3+1}$$

Be sure to check that all appropriate conditions hold.

Solution:

$$f(x) = \frac{x^2}{x^3+1} \Rightarrow f'(x) = \frac{2x(x^3+1) - x^2 \cdot 3x^2}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2}$$

$$= \frac{x(2-x^3)}{(x^3+1)^2} < 0 \leftarrow \text{on } [2,\infty)$$

$f(x)$ continuous and positive on $[1,\infty) \Rightarrow$ Can apply I.T.

Let $u = x^3+1 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$

$$\Rightarrow \int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+1| + C$$

$$\Rightarrow \int_1^{\infty} \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \frac{1}{3}\ln|t^3+1| - \frac{1}{3}\ln(2) = \infty$$

$\underset{\text{I.T.}}{\Rightarrow} \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ divergent

However, $\left\{\frac{n^2}{n^3+1}\right\}$ (eventually) decreasing and $\lim_{n\to\infty}\frac{n^2}{n^3+1} = 0$

$\underset{\text{A.S.T.}}{\Rightarrow} \sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ conditionally conv

4. (30 points) Find a power series (centered at 2) that represents the following function on an open interval containing 2.

$$f(x) = \frac{x-2}{(3-x)^3}$$

Carefully justify your answer and be sure to include a general term.

What is the value of $f^{(2022)}(2)$?

Solution:

$|x-2|<1 \Leftrightarrow x$ in $(1,3)$

$$\frac{1}{3-x} = \frac{1}{1-(x-2)} = 1 + (x-2) + (x-2)^2 + \dots$$

$$\downarrow \frac{d}{dx}$$

$$\frac{1}{(3-x)^2} = 1 + 2(x-2) + 3(x-2)^2 + \dots$$

$$\downarrow \frac{d}{dx}$$

$$\frac{2}{(3-x)^3} = 2 + 3\cdot 2(x-2) + 4\cdot 3(x-2)^2 + \dots$$

$$\downarrow \cdot (x-2)$$

$(1,3)$

$$\frac{(x-2)}{(3-x)^3} = \frac{2}{2}(x-2) + \frac{3\cdot 2}{2}(x-2)^2 + \dots + \frac{(n+1)n}{2}(x-2)^n + \dots$$

$$\Rightarrow c_{2022} = \frac{2023\cdot 2022}{2}$$

$$\Rightarrow \frac{f^{(2022)}(2)}{2022!} = \frac{2023\cdot 2022}{2} \Rightarrow f^{(2022)}(2) = \frac{2023\cdot 2022\cdot 2022!}{2}$$

5. (30 points) Show that the polynomial

$$\frac{x}{2} - \frac{x^2}{8}$$

approximates the function $f(x) = \ln(1 + \frac{x}{2})$ to within $\frac{1}{3}$ for all x in $[-1, 1]$. Carefully justify your answer.

Solution:

$$\ln(1+x) \overset{(-1,1)}{=} x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \Rightarrow \ln\left(1+\frac{x}{2}\right) \overset{(-2,2)}{=} \underbrace{\overbrace{\frac{x}{2} - \frac{x^2}{2^2 \cdot 2}}^{T_2(x)} + \frac{x^3}{2^3 \cdot 3} \cdots}_{\text{Maclaurin Series}}$$

$$f(x) = \ln\left(1+\frac{x}{2}\right) \Rightarrow f'(x) = \frac{1}{2}\,\frac{1}{1+\frac{x}{2}}$$

$$\Rightarrow f''(x) = (-1)\left(\frac{1}{2}\right)^2 \cdot \frac{1}{\left(1+\frac{x}{2}\right)^2} \Rightarrow f'''(x) = (-1)^2\, 2!\left(\frac{1}{2}\right)^3 \frac{1}{\left(1+\frac{x}{2}\right)^3}$$

Take max value at $x=-1$ on $[-1,1]$

$$\Rightarrow |f'''(x)| \underset{\text{on } [-1,1]}{\leq} 2! \cdot \left(\frac{1}{2}\right)^3 \cdot \frac{1}{\left(\frac{1}{2}\right)^3} = 2!$$

$$\Rightarrow \left| \ln\left|1+\frac{x}{2}\right| - \left(\frac{x}{2} - \frac{x^2}{8}\right) \right| \leq \frac{2!\,|x|}{3!} \leq \frac{1}{3}$$

Taylor's Inequality

on $[-1,1]$

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