MATH 1B MIDTERM 1 (LEC 001)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{M_n}| \leq \frac{K(b-a)^3}{24n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

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Student ID: ______________________________

GSI's name: ______________________________

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This exam consists of 5 questions. Answer the questions in the spaces provided.

1. (30 points) Compute the following integrals:

(a)

$$\int \tan^3(x)\sec^3(x)dx$$

Solution:

$$\int \tan^3(x)\sec^3(x)\,dx = \int \sin^3(x)\cos^{-6}(x)\,dx$$

Let $u = \cos(x) \Rightarrow \frac{du}{d} = -\sin(x) \Rightarrow dx = \frac{-du}{\sin(x)}$

$$\Rightarrow \int \sin^3(x)\cos^{-6}(x)\,dx = -\int (1-u^2)\,u^{-6}\,du = \frac{1}{5}u^{-5} - \frac{1}{3}u^{-3} + C$$

$$= \frac{1}{5}\cos^{-5}(x) - \frac{1}{3}\cos^{-3}(x) + C$$

(b)

$$\int x^3\sin(x^2)\ dx$$

Solution:

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$$\Rightarrow \int x^3\sin(x^2)\,dx = \frac{1}{2}\int u\sin(u)\,du$$

u $\sin(u)$

1 $-\cos(u)$

$$= -\frac{1}{2}u\cos(u) + \frac{1}{2}\int \cos(u)\,du = -\frac{1}{2}u\cos(u) + \frac{1}{2}\sin(u) + C$$

$$= -\frac{1}{2}x^2\cos(x^2) + \frac{1}{2}\sin(x^2) + C$$

2. (30 points) By doing a suitable **trigonometric substitution**, compute the following integral
$$\int \frac{x^5}{(\sqrt{4x^2-1})^3}dx$$
Carefully justify your answer. You should simplify your final answer using an appropriate right triangle.

Solution:

$$x = \frac{\sec(\theta)}{2} \Rightarrow \frac{dx}{d\theta} = \frac{1}{2}\tan(\theta)\sec(\theta) \Rightarrow dx = \frac{1}{2}\tan(\theta)\sec(\theta)\,d\theta$$

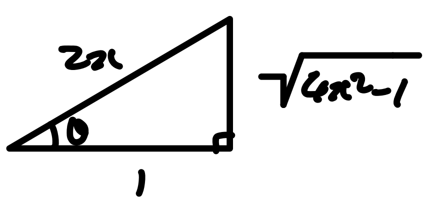
$$\Rightarrow \int \frac{x^5}{(\sqrt{4x^2-1})^3}dx = \frac{1}{2^6}\int \frac{\sec^5(\theta)}{\tan^3(\theta)}\tan(\theta)\sec(\theta)\,d\theta = \frac{1}{2^6}\int \tan^{-2}(\theta)\sec^6(\theta)\,d\theta$$

$$u = \tan(\theta) \Rightarrow \frac{du}{d\theta} = \sec^2(\theta) \Rightarrow d\theta = \frac{du}{\sec^2(\theta)}$$

$$\Rightarrow \int \tan^{-2}(\theta)\sec^6(\theta)\,d\theta = \int u^{-2}(1+u^2)^2\,du = \int u^{-2}+2+u^2\,du$$

$$= -u^{-1}+2u+\frac{1}{3}u^3+C = -\tan^{-1}(\theta)+2\tan(\theta)+\frac{1}{3}\tan^3(\theta)+C$$

$$x = \frac{\sec(\theta)}{2} \Rightarrow 2x = \sec(\theta) \Rightarrow \cos(\theta) = \frac{1}{2x}$$



$$\Rightarrow \tan(\theta) = \sqrt{4x^2-1}$$

$$\Rightarrow \int \frac{x^5}{(\sqrt{4x^2-1})^3}dx = \frac{1}{2^6}\left(-(\sqrt{4x^2-1})^{-1}+2\sqrt{4x^2-1}+\frac{1}{3}(\sqrt{4x^2-1})^3\right)+C$$

3. (30 points) (a) Write down the general form of the partial fraction decomposition of the function

$$\frac{r(x)}{(x^2+1)^2(x+1)}$$

where $r(x)$ is a polynomial with $\deg(r(x)) < 5$.

Solution:

$$\frac{r(x)}{(x^2+1)^2(x+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x+1}$$

(b) In the case that $r(x) = x^4+4x^2+2x+1$, show that function can actually be written as

$$\frac{A}{x+1} + \frac{Bx}{(x^2+1)^2}.$$

Use this to compute the integral

$$\int \frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)}dx$$

Solution:

$$\frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)} = \frac{A}{x+1} + \frac{Bx}{(x^2+1)^2} = \frac{A(x^2+1)^2 + Bx(x+1)}{(x+1)(x^2+1)^2}$$

$$= \frac{Ax^4+(2A+B)x^2+Bx+A}{(x+1)(x^2+1)^2}$$

$$\Rightarrow \begin{array}{l} A=1 \\ 2A+B=4 \\ B=2 \\ A=1 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=2 \end{array}$$

$$\Rightarrow \int \frac{x^4+4x^2+2x+1}{(x^2+1)^2(x+1)}dx = \int \frac{1}{x+1}dx + \int \frac{2x}{(x^2+1)^2}dx$$

$$= \ln|x+1| - \frac{1}{x^2+1} + C$$

4. (30 points) Determine if the following improper integrals are **convergent or divergent**. Justify your answers.

(a)

$$\int_{-\infty}^{\infty} \frac{x}{(x^2-1)^3} dx$$

$x^2 - 1 = 0 \Leftrightarrow x = \pm 1$

Solution:

II/2

Consider $\int_1^2 \frac{x}{(x^2-1)^3} dx$

If divergent then so is $\int_{-\infty}^{\infty} \frac{x}{(x^2-1)^3} dx$

$u = x^2 - 1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow \int \frac{x}{(x^2-1)^3} dx = \frac{1}{2}\int \frac{1}{u^3} du$

$= \frac{-1}{4}\frac{1}{u^2} + C$

$= \frac{-1}{4(x^2-1)^2} + C$

$\int_1^2 \frac{x}{(x^2-1)^3} dx = \lim_{t \to 1^+} \frac{-1}{4(x^2-1)^2}\Big|_t^2 = \lim_{t \to 1^+} \frac{-1}{4 \cdot 9} + \frac{1}{4(t^2-1)^2}$

$= \infty \Rightarrow$ <u>divergent</u>

(b)

$$\int_0^{\infty} \frac{\cos(x) + \sin(x) + 3}{x^3 + x^2 + 2x + 1} dx$$

Solution:

definite integral does not affect convergence

$\int_0^{\infty} \frac{\cos(x) + \sin(x) + 3}{x^3 + x^2 + 2x + 1} dx = \int_0^{1} \frac{\cos(x) + \sin(x) + 3}{x^3 + x^2 + 2x + 1} dx + \int_1^{\infty} \frac{\cos(x) + \sin(x) + 3}{x^3 + x^2 + 2x + 1} dx$ I/

$0 < \cos(x) + \sin(x) + 3 < 5$ <u>and</u> $0 < \frac{1}{x^3 + x^2 + 2x + 1} < \frac{1}{x^3}$ on $[1, \infty)$

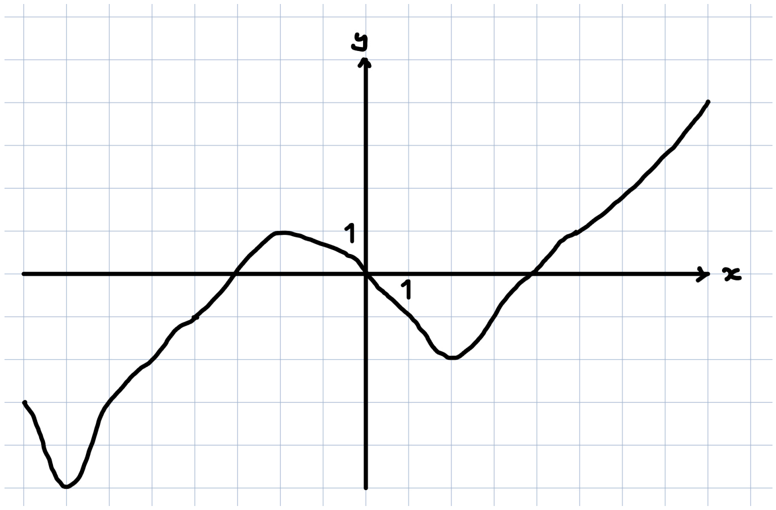
$\Rightarrow 0 < \frac{\cos(x) + \sin(x) + 3}{x^3 + x^2 + 2x + 1} < \frac{5}{x^3}$ on $[1, \infty)$

$\int_1^{\infty} \frac{5}{x^3} dx = \lim_{t \to \infty} \frac{-5}{2x^2}\Big|_1^t = \lim_{t \to \infty} \frac{-5}{2t^2} + \frac{5}{2} = \frac{5}{2}$

$\Rightarrow$ <u>convergent</u>

C.T.

5. (30 points) Let f be a function whose **fourth derivative** has the following graph.



Consider the integral $\int_{-4}^{4} f(x)dx$.

Assume that the *eighth* Simpson's approximation to this integral is $S_8 = \frac{93}{45}$.

Given this information, is it possible that $\int_{-4}^{4} f(x)dx = \frac{98}{45}$? Carefully justify your answer.

Solution:

Graph $\Rightarrow$ $|f^{(4)}(x)| \leq 2$ for all x in $[-4,4]$

$$\Rightarrow \quad |E_{S_8}| \leq \frac{2(4-(-4))^5}{180 \cdot 8^4} = \frac{16}{180} = \frac{4}{45}$$

However if $\int_{-4}^{5} f(x)dx = \frac{98}{45}$ $\Rightarrow$ $|E_{S_8}| = \left|\frac{98}{45} - \frac{93}{45}\right| = \frac{5}{45}$

and $\frac{5}{45} > \frac{4}{45}$

Conclusion : No! $\int_{-4}^{5} f(x)dx \neq \frac{98}{45}$

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