Math 1A. Solutions to Second Midterm

1. (16 points) Find the following derivatives.

(a). $\dfrac{d}{dx}\left(e^x \cos x\right)$

By the Product Rule: $e^x \cos x - e^x \sin x$.

(b). $\dfrac{d}{dx}\left(\dfrac{\sin x}{x^2+1}\right)$

By the Quotient Rule: $\dfrac{(x^2+1)\cos x - 2x\sin x}{(x^2+1)^2}$

(c). $\dfrac{d}{dx}\left(x^{\arctan x}\right)$

Let $y = x^{\arctan x}$. Then $\ln y = (\arctan x)\ln x$. Taking derivatives of both sides (and noting that you need to use the Chain Rule when differentiating the left-hand side) gives:

$$\frac{y'}{y} = \frac{\ln x}{x^2+1} + \frac{\arctan x}{x} \;,$$

so we have

$$\frac{d}{dx}\left(x^{\arctan x}\right) = y' = y\left(\frac{\ln x}{x^2+1} + \frac{\arctan x}{x}\right) = x^{\arctan x}\left(\frac{\ln x}{x^2+1} + \frac{\arctan x}{x}\right) \;.$$

(d). $\dfrac{d}{dx}\left(\tan\sqrt{x^2+1}\right)$

This is the composite of the three functions $\tan x$, $\sqrt{x}$, and x^2+1 . Therefore, we apply the Chain Rule twice to get

$$\frac{d}{dx}\left(\tan\sqrt{x^2+1}\right) = \left(\sec^2\sqrt{x^2+1}\right)\cdot\frac{1}{2\sqrt{x^2+1}}\cdot 2x = \frac{x\sec^2\sqrt{x^2+1}}{\sqrt{x^2+1}} \;.$$

2. (12 points) If $x = y - \cos y$, find y' and y'' at the point where $y = \pi/4$.

First: When $y = \pi/4$, we have $x = \pi/4 - \cos\pi/4 = \pi/4 - \sqrt{2}/2$.
Next, by implicit differentiation, we have

$$1 = y' + \sin y \cdot y' \;;$$
$$y' = \frac{1}{1+\sin y} \;.$$

When $y = \pi/4$, we then have

$$y' = \frac{1}{1+\sqrt{2}/2} \;.$$

Finally, taking the derivative of $y' = 1/(1+\sin y)$ gives

$$y'' = -\frac{\cos y \cdot y'}{(1+\sin y)^2}\ ,$$

so when $y = \pi/4$ we have

$$y'' = -\frac{\frac{\sqrt{2}/2}{1+\sqrt{2}/2}}{(1+\sqrt{2}/2)^2} = -\frac{\sqrt{2}/2}{(1+\sqrt{2}/2)^3} = -\frac{2}{(\sqrt{2}+1)^3}\ .$$

3. (10 points) Use logarithmic differentiation to find the derivative of the function

$$y = \frac{1-2x}{x^3(x-1)^4\sqrt{x^2+7}}\ .$$

You do not need to simplify your answer.

For full credit, use logarithmic differentiation to the greatest extent reasonably possible.

Take the logarithm of the absolute value of both sides of the above equation, and then apply the logarithm property:

$$\begin{aligned} \ln|y| &= \ln\left|\frac{1-2x}{x^3(x-1)^4\sqrt{x^2+7}}\right| \\ &= \ln|1-2x| - 3\ln|x| - 4\ln|x-1| - \frac{1}{2}\ln(x^2+7)\ . \end{aligned}$$

Now take the derivative of both sides, remembering to use the Chain Rule whenever differentiating anything involving y. This gives

$$\frac{y'}{y} = \frac{-2}{1-2x} - \frac{3}{x} - \frac{4}{x-1} - \frac{2x}{2(x^2+7)}\ .$$

Now solve for y':

$$\begin{aligned} y' &= y\left(\frac{-2}{1-2x} - \frac{3}{x} - \frac{4}{x-1} - \frac{x}{x^2+7}\right) \\ &= \frac{1-2x}{x^3(x-1)^4\sqrt{x^2+7}}\left(\frac{-2}{1-2x} - \frac{3}{x} - \frac{4}{x-1} - \frac{x}{x^2+7}\right)\ . \end{aligned}$$

(This does not need to be simplified any further.)

4. (12 points) A particle moves along the elliptical path given by $4x^2 + y^2 = 4$ in such a way that when it is at the point $(\sqrt{3}/2, 1)$, its x-coordinate is increasing at the rate of 5 units per second. How fast is the y-coordinate changing at that instant?

Use the method of related rates.

First, differentiate both sides of $4x^2 + y^2 = 4$, being sure to use the Chain Rule when differentiating anything involving *either variable* x or y, and then solve for y':

$$\begin{aligned} 8xx' + 2yy' &= 0 ; \\ 2yy' &= -8xx' ; \\ y' &= -\frac{8xx'}{2y} = -\frac{4xx'}{y} . \end{aligned}$$

Substituting the values $x = \sqrt{3}/2$, $y = 1$, and $x' = 5$ gives

$$y' = -\frac{4(\sqrt{3}/2)\cdot 5}{1} = -10\sqrt{3} \text{ units/second} .$$

5. (12 points) Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{29 - x^2}$ for all $x \in [0, 5]$.

(a). Use a linear approximation to estimate $g(1.95)$ and $g(2.05)$.

We'll use the linear approximation for $g(x)$ at $x = 2$. Since $g(2) = -4$ and $g'(2) = \sqrt{29-4} = \sqrt{25} = 5$, the linear approximation is

$$L(x) = -4 + 5(x - 2) ,$$

so

$$g(1.95) \approx L(1.95) = -4 + 5(-.05) = -4 - .25 = -4.25$$

and

$$g(2.05) \approx L(2.05) = -4 + 5(.05) = -4 + .25 = -3.75 .$$

This can also be done by letting $dx = \Delta x = \pm .05$, so then

$$\Delta y \approx dy = f'(2)\, dx = 5(\pm .05) = \pm .25 .$$

(b). Is your estimate for $g(2.05)$ in part (a) too large or too small? Explain.

Since $g'(x)$ is a decreasing function of x on $[0, 5]$, $g'(x) < g'(2)$ for all such $x \in (2, 2.05)$. Therefore $g(x)$ is increasing more slowly than $L(x)$ to the right of $x = 2$, so $g(2.05) < L(2.05)$, and the estimate is too high.

Likewise, $g'(x) > g'(2)$ when $x \in (1.95, 2)$. Therefore $g(x)$ is increasing more quickly than $L(x)$ to the left of $x = 2$. So, again, $g(1.95) < L(1.95)$ (since g catches up to L as $x \to 2^-$). Therefore the estimate for $g(1.95)$ is also too high.

Or, one can also note that

$$g''(x) = \frac{-2x}{2\sqrt{29 - x^2}} \ ,$$

and that this is negative on $(0,5)$. Therefore the graph of $g(x)$ is concave down on $(0,5)$. This means that the graph of $g(x)$ is below the tangent line $y = L(x)$, so $L(x)$ is above $g(x)$, and both estimates are too high.

[This is a variation of Exercise 44 on page 257, which was one of the Supplementary Exercises for Week 9.]

6. (12 points) Determine the maximum and minimum values of the function

$$f(x) = x - 3x^{2/3}$$

on the interval $[-1, 27]$, and find all values of x (in $[-1, 27]$) where they occur.

We use the Closed Interval Method.
First, look for critical points:

$$f'(x) = 1 - 2x^{-1/3} = 1 - \frac{2}{\sqrt[3]{x}} \ .$$

This does not exist when $x = 0$, and is zero when $2 = \sqrt[3]{x}$, so $x = 8$. Computing the values of f at these numbers gives:

$$f(0) = 0$$
$$f(8) = 8 - 3 \cdot 4 = 8 - 12 = -4 \ .$$

Next, we evaluate f at the endpoints of the interval:

$$f(-1) = -1 - 3(-1)^{2/3} = -1 - 3\sqrt[3]{1} = -1 - 3 = -4$$
$$f(27) = 27 - 3(27)^{2/3} = 27 - 3(\sqrt[3]{27})^2 = 27 - 3(3^2) = 27 - 27 = 0 \ .$$

Therefore the maximum value is 0, which occurs at $x = 0$ and $x = 27$, and the minimum value is -4, which occurs at $x = 8$ and $x = -1$.

7. (14 points) Compute the following limits. Give the value if the limit exists, or if it is ∞ or $-\infty$. If none of these are true, write "No limit."

(a). $\lim_{x \to 0} \dfrac{\sin x - x}{x^3}$

Use l'Hospital's Rule three times:

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6} \ .$$

(b). $\lim_{x \to \infty} \left(1 - \dfrac{3}{x}\right)^{2/x}$

This is the determinate form $1^0 = 1$ (or, its log is of the determinate form $0 \ln 1 = 0 \cdot 0$). So the limit equals 1.

(c). $\lim_{x\to\infty} \dfrac{3x+2}{\sqrt{x^2+7}}$

L'Hospital's Rule doesn't help with this one, so instead we divide the numerator and denominator by x to get

$$\lim_{x\to\infty} \frac{3x+2}{\sqrt{x^2+7}} = \lim_{x\to\infty} \frac{3+2/x}{\sqrt{1+7/x^2}} = \frac{3+0}{\sqrt{1+0}} = 3 \ .$$

8. (12 points) Find an equation of the slant asymptote as $x \to \infty$. (Do not sketch the curve.)

$$y = \sqrt{x^2 + 6x + 7/x}$$

We follow the method given in class on October 26.

To get the slope of the slant asymptote (if there is one), we use

$$m = \lim_{x\to\infty} \frac{\sqrt{x^2+6x+7/x}}{x} = \lim_{x\to\infty} \sqrt{1+6/x+7/x^3} = \sqrt{1+0+0} = 1 \ ,$$

so $m = 1$. Then, to get the y-intercept, we use

$$\begin{aligned}
b &= \lim_{x\to\infty} \left(\sqrt{x^2+6x+7/x} - mx\right) \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2+6x+7/x} - x}{1} \\
&= \lim_{x\to\infty} \frac{(\sqrt{x^2+6x+7/x} - x)(\sqrt{x^2+6x+7/x} + x)}{\sqrt{x^2+6x+7/x} + x} \\
&= \lim_{x\to\infty} \frac{x^2+6x+7/x - x^2}{\sqrt{x^2+6x+7/x} + x} \\
&= \lim_{x\to\infty} \frac{6x+7/x}{\sqrt{x^2+6x+7/x} + x} \\
&= \lim_{x\to\infty} \frac{6+7/x^2}{\sqrt{1+6/x+7/x^3} + 1} \\
&= \frac{6+0}{\sqrt{1+0+0}+1} \\
&= \frac{6}{2} = 3 \ .
\end{aligned}$$

Therefore the curve has a slant asymptote, which is the line $y = x + 3$.

Notes.

(1). Many students tried to do this by long division. (Long division only works for rational functions.)

(2). Quite a few students misinterpreted $x^2 + 6x + 7/x$ as $(x^2 + 6x + 7)/x$ ($x^2 + 6x + (7/x)$ was correct).