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MATH 53 1st MIDTERM

Please answer each question on a separate page – you can write on the back of the page. Remember to write your name and section number on EVERY page. Each problem is worth 10 points. Good Luck!

Problem 1. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $\mathbb{R}^3$.

a) Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 0$. (You can use facts from the book/lectures as long as you *state them clearly.*)

We now from the properties of the cross product and vector addition that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}.$$

And we proved in class that

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0.$$

The conclusion follows.

b) if $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard basis vectors, find

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{k}) + \mathbf{j} \cdot (\mathbf{a} \times \mathbf{i}) + \mathbf{k} \cdot (\mathbf{a} \times \mathbf{j})$$

in terms of a_1, a_2, a_3.

We recall the defintion (for $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$)

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle, \tag{1}$$

so that, since $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$,

$$\mathbf{a} \times \mathbf{k} = \langle a_2, -a_1, 0 \rangle, \quad \mathbf{a} \times \mathbf{j} = \langle -a_3, 0, a_1 \rangle, \quad \mathbf{a} \times \mathbf{i} = \langle 0, a_3, -a_2 \rangle.$$

Hence,

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{k}) = a_2, \quad \mathbf{j} \cdot (\mathbf{a} \times \mathbf{i}) = a_3, \quad \mathbf{k} \cdot (\mathbf{a} \times \mathbf{j}) = a_1,$$

and

$$\mathbf{i} \cdot (\mathbf{a} \times \mathbf{k}) + \mathbf{j} \cdot (\mathbf{a} \times \mathbf{i}) + \mathbf{k} \cdot (\mathbf{a} \times \mathbf{j}) = a_1 + a_2 + a_3.$$

c) Find the area of the *parallelogram* spanned by $\langle 1, 2, 3 \rangle$ and $\langle -1, 2, 3 \rangle$ (use the back page if needed).

The area of the parallelogram is given by the length of the cross product of the vectors defining the parallelogram. Hence, using

$$\begin{aligned} \text{Area} &= |\langle 1, 2, 3 \rangle \times \langle -1, 2, 3 \rangle| = |\langle 2 \cdot 3 - 3 \cdot 2, 3 \cdot (-1) - 1 \cdot 3, 1 \cdot 2 - 2 \cdot (-1) \rangle| \\ &= |\langle 0, -6, 4 \rangle| = \sqrt{0 + 36 + 16} = \sqrt{52} = \sqrt{4 \cdot 13} \\ &= 2\sqrt{13}. \end{aligned}$$

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Problem 2. a) Show that of the two curves parametrized by position vectors

$$\mathbf{r}_1(t) = \langle \cos(2\pi t), \sin(2\pi t), t \rangle, \qquad \mathbf{r}_2(t) = \langle t-1, t, t/4 \rangle. \tag{2}$$

intersect at the point $(0, 1, 1/4)$ and find $\cos\theta$ where θ is the angle between the tangent vectors to the two curves at that point.

From the expressions for $\mathbf{r}_1$ and $\mathbf{r}_2$ we see that for the last component to be $\frac{1}{4}$ we have to take $t = \frac{1}{4}$ for $\mathbf{r}_1$ and $t = 1$ for $\mathbf{r}_2$. We then check that

$$\mathbf{r}_1(\tfrac{1}{4}) = \langle \cos(\pi/2), \sin(\pi/2), \tfrac{1}{4} \rangle = \langle 0, 1, \tfrac{1}{4} \rangle, \quad \mathbf{r}_2(1) = \langle 0, 1, \tfrac{1}{4} \rangle.$$

We compute tangent vectors by differentiating the position vectors:

$$\mathbf{r}_1'(t) = \langle -2\pi \sin(2\pi t), 2\pi \cos(2\pi t), 1 \rangle, \quad \mathbf{r}_2'(t) = \langle 1, 1, \tfrac{1}{4} \rangle.$$

The tangent vectors at the point of intersection are given by

$$\mathbf{r}_1'(\tfrac{1}{4}) = \langle -2\pi, 0, 1 \rangle, \quad \mathbf{r}_2'(1) = \langle 1, 1, \tfrac{1}{4} \rangle.$$

Hence

$$\cos\theta = \frac{\mathbf{r}_1'(\frac{1}{4}) \cdot \mathbf{r}_2'(1)}{|\mathbf{r}_1'(\frac{1}{4})||\mathbf{r}_2'(1)|} = \frac{-2\pi + \frac{1}{4}}{(4\pi^2+1)^{\frac{1}{2}}(2+\frac{1}{16})^{\frac{1}{2}}} = \frac{-8\pi + 1}{\sqrt{33}(4\pi^2+1)^{\frac{1}{2}}}.$$

b) Is there a value of t for which $\mathbf{r}_1'(t)$ and $\mathbf{r}_2'(t)$ are parallel? (Please use the back page if you need more space.)

For $\mathbf{r}_1'(t)$ and $\mathbf{r}_2'(t)$ to be parallel we need, for some scalar c,

$$\langle -2\pi \sin(2\pi t), 2\pi \cos(2\pi t), 1 \rangle = c \langle 1, 1, \tfrac{1}{4} \rangle.$$

That means that $c = 4$ and

$$-2\pi \sin(2\pi t) = 2\pi \cos(2\pi t) = 4.$$

But $-\sin(2\pi t) = \cos(2\pi t)$ only if $2\pi t = 3\pi/4 + k\pi$. But then the value of $2\pi\cos(2\pi t) = \pm 2\pi/\sqrt{2} \neq 4$.

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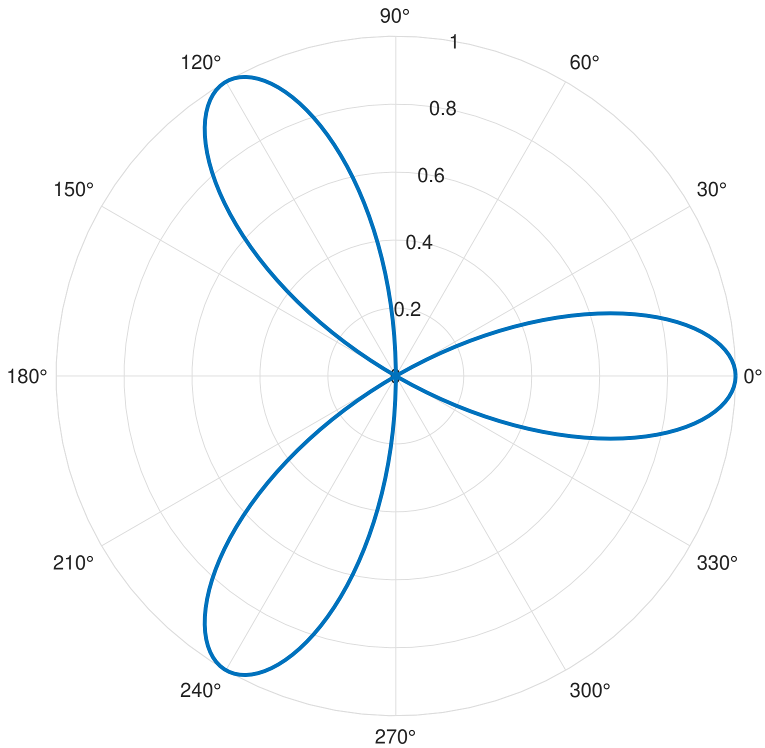
Problem 3. a) Sketch the polar curve given by the equation $r = \cos(3\theta)$, $0 \le \theta \le \pi$.



b) Compute the area enclosed by this curve. (Please use the other side of the page if needed.)

The area of the region defined by a polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, is given by $\frac{1}{2}\int_\alpha^\beta f(\theta)^2 d\theta$. For $f(\theta) = \cos 3\theta$, $\alpha = 0$, $\beta = \pi$, gives

$$\text{Area} = \frac{1}{2}\int_0^\pi \cos^2(3\theta) d\theta = \frac{1}{6}\int_0^{3\pi} \cos^2 t dt.$$

Note that

$$\int_0^{3\pi} \cos^2 t dt = \int_0^{3\pi} \sin^2 t dt$$

and hence

$$\int_0^{3\pi} \cos^2 t dt = \frac{1}{2}\int_0^{3\pi} (\cos^2 t + \sin^2 t) dt = \frac{1}{2}\int_0^{3\pi} 1 dt = \frac{3}{2}\pi.$$

Hence

$$\text{Area} = \frac{1}{6}\frac{3}{2}\pi = \frac{1}{4}\pi.$$

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Problem 4. Match the following three dimensional curves to their equations. The parameter satisfies $0 \leq t \leq \pi$ for all curves.

a) $\mathbf{r}(t) = \langle t^3, t, t^3 \rangle$

b) $\mathbf{r}(t) = \langle t, t^3, t^3 \rangle$

c) $\mathbf{r}(t) = \langle \sin(t/2), t^3, t^3 \rangle$

d) $\mathbf{r}(t) = \langle t^3 \cos(t/2), t, t^3 \rangle$

Please do not guess: negative points will be given for wrong matches. We have three versions of the exam with different arrangements of answers!

Please use scratch paper and record your answers in the table below:

Figure 1	**Figure 2**	**Figure 3**	**Figure 4**
b	**c**	**a**	**d**

Figure 1

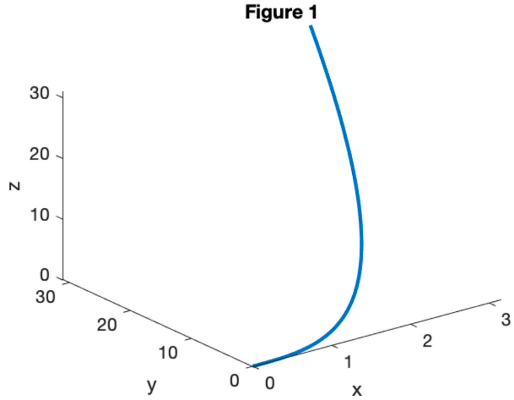


Figure 2

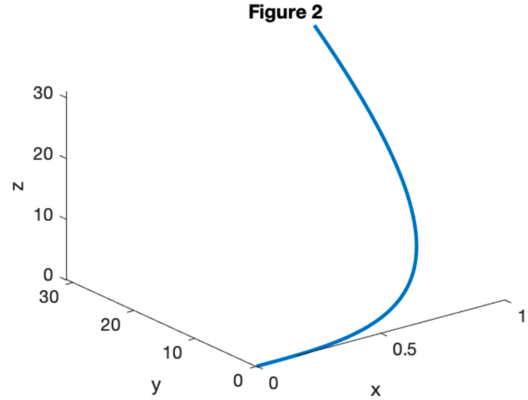


Figure 3

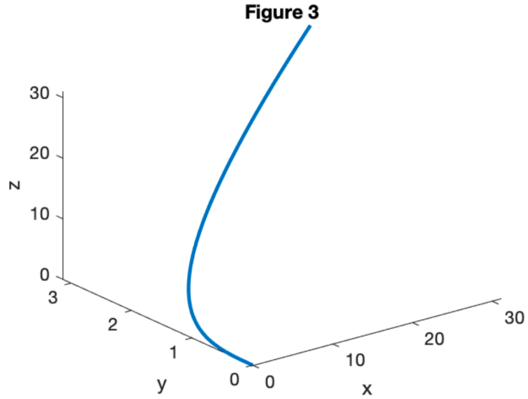
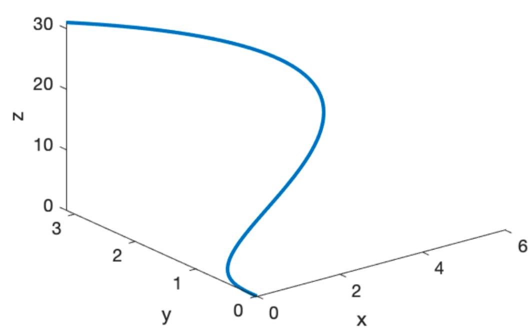


Figure 4



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Problem 5. a) Find the tangent plane to the graph of the function

$$f(x, y) = x^3 + 2x^2y + 2$$

at the point $(x_0, y_0, f(x_0, y_0))$ where $(x_0, y_0) = (1, -1)$.

The tangent plane to the graph of f at (x_0, y_0, z_0), $z_0 = f(x_0, y_0)$, is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

In our case $x_0 = 1$, $y_0 = -1$ and $z_0 = 1^3 + 2 \cdot 1^2 \cdot (-1) + 2 = 1$. The partial derivatives are given by

$$f_x(x, y) = 3x^2 + 4xy, \quad f_y(x, y) = 2x^2,$$

so

$$f_x(1, -1) = -1, \quad f_y(1, -1) = 2.$$

The tangent plane is then

$$z - 1 = -(x - 1) + 2(y + 1) \text{ or } x - 2y + z - 4 = 0.$$

b) What is the distance of the point $(1, 2, 3)$ to the plane you found in part a).

The distance of a point (x_1, y_1, z_1) to the plane defined by $ax + by + cz + d = 0$ is given by

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

In our case we found above that $a = 1$, $b = -2$, $c = 1$ and $d = -4$. Hence the distance of $(1, 2, 3)$ to the tangent plane is given by

$$\frac{|1 - 4 + 3 - 4|}{\sqrt{1 + 4 + 1}} = \frac{4}{\sqrt{6}}.$$