EECS 126: Probability and Random Processes

Solutions to Problem Set 10 (mid-term 2)

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Problem 1

a) False.

A counterexample: let X and B be independent random variables, $X \sim N(0,1)$, B is a Bernoulli random variable where $P(B=-1) = P(B=1) = 0.5$. Let $Y = BX$, obviously X and Y are not independent, $X \sim N(0,1)$ and $E(XY) = E(X^2 B) = E(X^2)E(B) = 0$.

b) False.

Proof by contradiction: suppose there exists a function g, s.t. $Y = g(X)$ is uniformly distributed in $[0,1]$. Then let $a = g(1)$,

$$Pr(Y = a) = Pr(g(X) = a) = Pr(g(X) = g(1)) \geq Pr(X = 1) > 0$$

It's impossible because Y is a uniform random variable.

c) True.

proof :

The MMSE estimation of X is $\hat{X} = E(X|Y)$, so

$$E(\tilde{X}) = E(X - \hat{X}) = E(X) - E(\hat{X}) = E(X) - E(E(X|Y)) = E(X) - E(X) = 0$$

$$\begin{aligned}
Cov(\tilde{X}, Y) &= E(\tilde{X}Y) - E(\tilde{X})E(Y) \\
&= E(\tilde{X}Y) \\
&= E((X - E(X|Y))Y) \\
&= E(XY) - E(YE(X|Y)) \\
&= E(XY) - E(E(XY|Y)) \\
&= E(XY) - E(XY) \\
&= 0
\end{aligned}$$

Problem 2

a) We will show that $X_i \sim N(0,1)$ using proof by induction.

proof : $X_0 \sim N(0,1)$, suppose $X_k \sim N(0,1)$. Then X_k and N_{k+1} have the same pdf

$$f_{X_k}(x) = f_{N_{k+1}}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

.

Then $\forall x \in \mathcal{R}$

$$\begin{aligned} f_{X_{k+1}}(x) &= f_{X_k}(x)P(B_{k+1}=0) + f_{N_{k+1}}(x)P(B_{k+1}=1) \\ &= p\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} + (1-p)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \end{aligned} \tag{1}$$

We just proved by induction that $X_i \sim N(0,1)$.

b) No.

Proof by contradiction, assume that X, Y are jointly continuous. Then $\exists$ pdf $f_{X_1X_2}(x_1,x_2)$, s.t. for every subset D of the two dimensional-plane:

$$P((X_1,X_2)\in D) = \int\int_{(x_1,x_2)\in D} f_{X_1X_2}(x_1,x_2)dx_1dx_2$$

Let $D = \{(x_1,x_2)\in \mathcal{R}^2 : x_1 = x_2\}$, then $P((X_1,X_2)\in D) = P(X_1 = X_2) = 0.5$

But if D is a set of area zero in $\mathcal{R}^2$, thus $\int\int_{(x_1,x_2)\in D} f_{X_1X_2}(x_1,x_2)dx_1dx_2 = 0$. Now we have a contradiction. Thus the origin assumption that X, Y are jointly continuous is false.

c) From part a), we know that $E(X_i) = 0$.

$$\begin{aligned} Cov(X_1,X_2) &= E(X_1X_2) - E(X_1)E(X_2) \\ &= E(E(X_1X_2|B_2)) \\ &= E(X_1X_2|B_2=0)P(B_2=0) + E(X_1X_2|B_2=1)P(B_2=1) \\ &= E(X_1^2)(1-p) + E(X_1N_2)p \\ &= (1-p) \end{aligned}$$

d) If $B_{i+1} = B_{i+2} = ... = B_j = 0$, then $X_j = X_i$, $E(X_iX_j) = 1$.
Otherwise $\exists l, \quad i+1 \le l \le j : B_l = 0$, let $k, \quad i+1 \le k \le j$ be the maximum index, s.t. $B_k = 1$, then $X_j = N_k$ and N_k and X_i are independent, $E(X_iX_j) = 0$. So:

$$\begin{aligned} Cov(X_i,X_j) &= E(X_iX_j) - E(X_i)E(X_j) \\ &= E(E(X_iX_j|B_{i+1},B_{i+2},...,B_j)) \\ &= E(X_iX_j|B_{i+1}=B_{i+2}=...=B_j=0)P(B_{i+1}=B_{i+2}=...=B_j=0) \\ &\quad +E(X_iX_j|\exists l, \quad i+1\le l\le j : B_l = 0)Pr(\exists l, \quad i+1\le l\le j : B_l=0) \\ &= E(X_i^2)(1-p)^{j-i} + 0 \\ &= (1-p)^{j-i} \end{aligned}$$

Problem 3

a) Condition on $X = x$, Y is uniformly distributed in $[x^2, x^2 + 1]$. Thus the conditional expectation of Y given $X = x$ is $x^2+0.5$. So the MMSE estimator is $\hat{X} = E(Y|X) = X^2+0.5$

b) Some of the statistics of X and Y are $E(X) = 0$,

$$E(X^2) = \int_{-0.5}^{0.5} x^2 dx = \frac{1}{12}$$

$$E(Y) = E(E(Y|X)) = E(X^2 + 0.5) = \frac{7}{12}$$

$$E(Y^2) = E(E(Y^2|X)) = \int_{-0.5}^{0.5} \int_{x^2}^{x^2+1} y^2 dy dx = \frac{7}{12}$$

The LLSE estimator is $\hat{Y} = aX + b$. Then

$$E((Y - \hat{Y})^2) \quad = \quad E((Y - aX - b)^2) = E(Y^2) + a^2 E(X^2) \tag{2}$$

c) The MMSE estimator $\hat{Y} = X^2 + 0.5$ has a quadratic form. So the best quadratic estimator (in mean square sense) is just the MMSE estimator $X^2 + 0.5$

d) X, Y are jointly Gaussian, so the MMSE estimator for Y given X is $aX + b$, thus the best quadratic estimator (in mean square sense) is just the MMSE estimator $aX + b$, so $a_3 = 0$.