MT2.1 (25 Points) Consider a continuous-time signal x defined as

$$x(t) = 10\,sinc\left(\frac{t+4}{7}\right),$$

where the *normalized sinc function* is defined as

$$sinc\,(t) = \frac{\sin(\pi t)}{\pi t}.$$

(a) Provide a well-labeled sketch of $X(\omega)$, the continuous-time Fourier transform of the signal x.

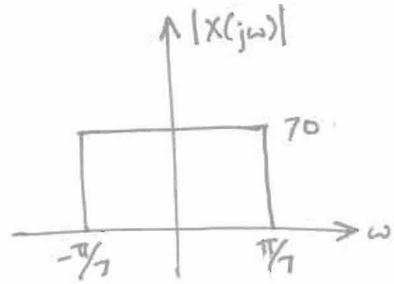
$$\text{sinc } t \overset{F}{\longleftrightarrow} \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$x(t-t_0) \overset{F}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega)$$

$$\Rightarrow 10\,\text{sinc}\left(\frac{t}{7}\right) \overset{F}{\longleftrightarrow} \begin{cases} 70, & |\omega| < \frac{\pi}{7} \\ 0, & \text{otherwise} \end{cases}$$

$$10\,\text{sinc}\left(\frac{t+4}{7}\right) \overset{F}{\longleftrightarrow} \begin{cases} 70e^{j4\omega}, & |\omega| < \frac{\pi}{7} \\ 0, & \text{otherwise} \end{cases}$$



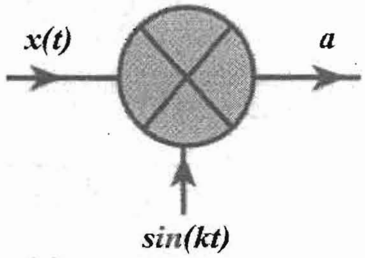
(b) Find the total area under the signal x – the quantity $\int_{-\infty}^{\infty} x(t)dt$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

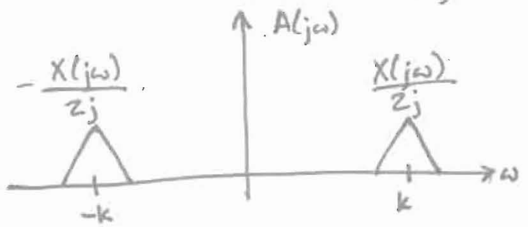
$$X(j0) = \int_{-\infty}^{\infty} x(t)\, dt$$

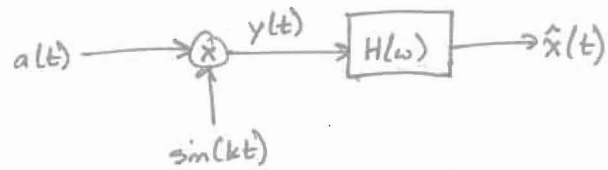
$$= 70e^{j4\omega}\Big|_{\omega=0} = 70$$

(c) In the figure below, the signal x modulates a carrier of frequency k rad/s. Sketch the spectra at point a. Finally, design a receiver to recover the signal x from the modulated signal at point a.



$$a(t) = x(t)\sin kt = x(t)\,\frac{e^{jkt} - e^{-jkt}}{2j}$$





$$y(t) = x(t)\sin^2(kt) = x(t)\left[\frac{e^{jkt} - e^{-jkt}}{2j}\right]^2$$

$$= x(t)\,\frac{e^{j2kt} - 2 + e^{-j2kt}}{-4}$$

Need $H(\omega)$ to be a LPF with gain of 2.

(d) Consider a continuous-time signal y defined as

$$y(t) = \begin{cases} t & |t| < 1 \\ 0 & elsewhere \end{cases}.$$

Determine $Y(\omega)$, the continuous-time Fourier transform of the signal y.

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t}\,dt$$

$$= \int_{-1}^{1} t e^{-j\omega t}\,dt$$

$$= t\,\frac{e^{-j\omega t}}{-j\omega} + \frac{e^{-j\omega t}}{\omega^2}\Bigg|_{t=-1}^{1}$$

$$= \frac{e^{-j\omega}}{-j\omega} + \frac{e^{j\omega}}{-j\omega} + \frac{e^{-j\omega}}{\omega^2} - \frac{e^{j\omega}}{\omega^2}$$

$$= -\frac{2}{j\omega}\cos\omega + \frac{2}{j\omega^2}\sin\omega$$

MT2.1d) Let $x(t) = \begin{cases} 1, & |t|<1 \\ 0, & \text{otherwise} \end{cases} \xleftrightarrow{F} X(j\omega) = 2\frac{\sin\omega}{\omega}$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \begin{cases} 0, & t<-1 \\ t+1, & -1<t<1 \\ 2, & 1<t \end{cases} \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

$$= \frac{2}{j\omega^2}\sin\omega + 2\pi\delta(\omega)$$

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau - u(t+1) - u(t-1)$$

$$u(t) \xleftrightarrow{F} \frac{1}{j\omega} + \pi\delta(\omega)$$

$$u(t+1) \xleftrightarrow{F} e^{j\omega}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$u(t-1) \xleftrightarrow{F} e^{-j\omega}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$u(t+1) + u(t-1) \xleftrightarrow{F} \frac{2}{j\omega}\cos\omega + 2\pi\delta(\omega)$$

$$Y(j\omega) = \frac{2}{j\omega^2}\sin\omega - \frac{2}{j\omega}\cos\omega$$

MT2.2 (15 Points) Consider a discrete-time FIR filter F having impulse response f and frequency response F. Each part below discloses partial information about the filter F. Ultimately, your task is to determine the impulse response f completely.

In the space provided for each part, state every inference that you can draw from the information disclosed up to, and including, that part.

Justify all your work succinctly, but clearly and convincingly.

(i) The impulse response f is even.

$f[n] = f[-n]$ $\qquad F\{f[-n]\} = \sum_{n=-\infty}^{\infty} f[-n]e^{-j\omega n}$

$= \sum_{m=-\infty}^{\infty} f[m]e^{j\omega m} = F(-\omega)$

(ii) The frequency response F is real-valued.

$F(\omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} f[n](\cos \omega n + j \sin \omega n)$

Real part of $f[n]$ even, imaginary part odd $\xRightarrow{(i)}$ $f[n]$ real

(iii) The LTI system characterized by the impulse response $g(n) = f(n-k)$ is causal if and only if $k > 1$.

$f[n] = 0$ for $n < -2$ $\qquad f[1] = f[-1]$

Since f even, $f[n] = 0$ for $|n| > 2$ $\qquad f[2] = f[-2]$

(iv) $\sum_{n=EvenIntegers} f(n) = 3$.

$f[-2] + f[0] + f[2] = \sum_{n=-2}^{2} \frac{(-1)^n + 1}{2} f[n] = 3$

(v) The filter's response to the input signal $x(n) = (-1)^n$ is $y(n) = 0$.

$(-1)^n = e^{j\pi n}$ $\qquad F(\pi) = 0$

$F(\pi) = \sum_{n=-2}^{2} f[n]e^{-j\pi n} = \sum_{n=-2}^{2} (-1)^n f[n] = 0 \xRightarrow{(iv)} \sum_{n=-2}^{2} f[n] = 6$

(vi) The filter's response to the input signal $x(n) = i^n$ is $y(n) = 2i^n$.

$i^n = e^{j\frac{\pi}{2}n}$ $\qquad F(\frac{\pi}{2}) = 2$

$F(\frac{\pi}{2}) = \sum_{n=-2}^{2} f[n]e^{-j\frac{\pi}{2}n} = -f[-2] + i f[-1] + f[0] - i f[1] - f[2] = 2$

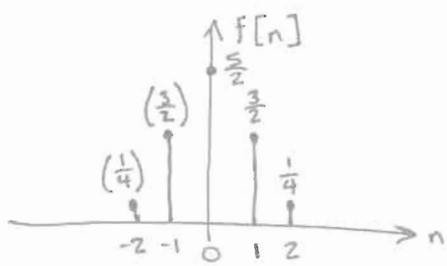
Determine, and provide a well-labeled plot of, the impulse response f.

(iv) $f[0] + 2f[2] = 3$

(v) $f[0] + 2f[1] + 2f[2] = 0$

(vi) $f[0] - 2f[2] = 2$

$$\Longrightarrow \begin{cases} f[0] = \frac{5}{2} \\ f[1] = f[-1] = \frac{3}{2} \\ f[2] = f[-2] = \frac{1}{4} \end{cases}$$



MT2.3 (25 Points) The reconstruction of a continuous-time signal y_c from its samples y_d can be conceptualized as an impulse generator followed by an ideal low-pass filter. In the time domain, this amounts to the *sinc interpolation* step defined as

$$y_c(t) = \sum_{n\in\mathbf{Z}} y_d(n)\mathrm{sinc}(\frac{t-nT_s}{T_s}), \quad t \in \mathbf{R}.$$

(a) Show that the set of shifted sinc functions $\{sinc\,(t-n), n \in Z\}$ is orthonormal:

$$<sinc(t\text{-}k), sinc(t\text{-}l)> = \delta(k-l)$$

You may wish to invoke the *orthogonality preserving property* of the CTFT,

$$\langle \phi_k, \phi_l \rangle = \delta(k-l) \Leftrightarrow \langle \Phi_k, \Phi_l \rangle = 2\pi\delta(k-l),$$

where Φ_k is the CTFT of the continuous-time signal ϕ_k, $\forall k$.

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t} \xrightarrow{F} \mathrm{rect}(\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

$$x(t-k) \xrightarrow{F} X(\omega)e^{-j\omega k}$$

$$\text{Let } \phi_k = \mathrm{sinc}(t-k) \xrightarrow{F} \Phi_k = \begin{cases} e^{-j\omega k}, & |\omega| < \pi \\ 0, & \text{otherwise} \end{cases}$$

Similarly for ϕ_ℓ

$$\langle \Phi_k, \Phi_\ell \rangle = \int_{-\infty}^{\infty} \Phi_k(\omega)\Phi_\ell^*(\omega)\,d\omega = \int_{-\pi}^{\pi} e^{-j\omega k} e^{j\omega \ell}\,d\omega$$

$$\text{If } k=\ell \quad = \int_{-\pi}^{\pi} 1\,d\omega = 2\pi$$

$$\text{If } k\neq\ell \quad = \int_{-\pi}^{\pi} e^{j\omega(\ell-k)}\,d\omega = 0$$

$$\langle \Phi_k, \Phi_\ell \rangle = 2\pi\delta[k-\ell] \Rightarrow \langle \phi_k, \phi_\ell \rangle = \delta[k-\ell]$$

It can be shown that any bandlimited signal *f(t)* can be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \mathrm{sinc}(t-n)$$

where $\alpha_n = \langle \mathrm{sinc}(t-n), f(t) \rangle = f(n)$

One generalized interpretation of the sampling theorem is in terms of orthonormal bases for bandlimited functions.

The *Whittaker-Shannon* interpolation formula given above is not implementable in real-time because it is not causal. In practice, an implementable (causal) non-ideal low-pass filter is used.

(b) Consider a continuous-time implementable filter with impulse response $\hat{h}$ and frequency response $\hat{H}$. Here, we seek to minimize the *mean-square error* in the reconstructed signal:

$$\varepsilon_t = \int_{-\infty}^{\infty} \left|\hat{h}(t) - h(t)\right|^2 dt$$

Suppose that we determine the *mean-square error* in the non-ideal low-pass filter:

$$\varepsilon_\omega = \int_{-\infty}^{\infty} \left|\hat{H}(\omega) - H(\omega)\right|^2 d\omega$$

Describe how ε_t can be determined from ε_ω.

Since $\hat{h}(t) - h(t) \xrightarrow{\text{CTFT}} \hat{H}(\omega) - H(\omega)$

By Parseval's

$$\varepsilon_t = \frac{1}{2\pi} \varepsilon_\omega$$

(c) Instead, consider a discrete-time implementable FIR filter with impulse response $\hat{f}$ and frequency response $\hat{F}$. Here, we seek to approximate the ideal low-pass filter,

$$F(\omega) = \begin{cases} 1 & |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\omega| \le \pi \end{cases},$$

by minimizing the *mean-square error* at the following discrete frequencies:

$$S = \left\{-\pi, -\frac{2\pi}{3}, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}.$$

Determine the least-squares solution for the optimal length-4 sequence

$$\hat{f} = \begin{pmatrix}\hat{f}(0) & \hat{f}(1) & \hat{f}(2) & \hat{f}(3)\end{pmatrix}.$$

NOTE: You may leave matrix operations unevaluated in the solution.

Want to minimize $\sum_{\omega \in S} \left|F(\omega) - \hat{F}(\omega)\right|^2 = \min_{\hat{\underline{f}}} \|\underline{G} - D\hat{\underline{f}}\|$

$$\hat{F}(\omega) = \sum_{n=0}^{3} \hat{f}[n] e^{-j\omega n} \qquad \hat{\underline{f}}^* = (D^T D)^{-1} D^T \underline{G}$$

$$\underline{G} = \begin{bmatrix} F(-\frac{2\pi}{3}) \\ F(-\frac{\pi}{3}) \\ F(0) \\ F(\frac{\pi}{3}) \\ F(\frac{2\pi}{3}) \\ F(\pi) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \hat{F}(-\frac{2\pi}{3}) \\ \hat{F}(-\frac{\pi}{3}) \\ \hat{F}(0) \\ \hat{F}(\frac{\pi}{3}) \\ \hat{F}(\frac{2\pi}{3}) \\ \hat{F}(\pi) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & e^{j\frac{2\pi}{3}} & e^{2j\frac{2\pi}{3}} & 1 \\ 1 & e^{j\frac{\pi}{3}} & e^{2j\frac{\pi}{3}} & -1 \\ 1 & 1 & 1 & 1 \\ 1 & e^{-j\frac{\pi}{3}} & e^{-2j\frac{\pi}{3}} & -1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-2j\frac{2\pi}{3}} & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \hat{f}[0] \\ \hat{f}[1] \\ \hat{f}[2] \\ \hat{f}[3] \end{bmatrix}}_{\hat{\underline{f}}}$$

MT2.4 (25 Points) Consider a discrete-time N-periodic signal $x_p(n)$ and the length-N signal $x(n)$ obtained by keeping the single period $[0, N-1]$ of $x_p(n)$. Denote $X_p(k)$ as the discrete Fourier series (DFS) of $x_p(n)$ and $X(\omega)$ as the discrete-time Fourier transform (DTFT) of $x(n)$. In this problem, use the following DFS equations:

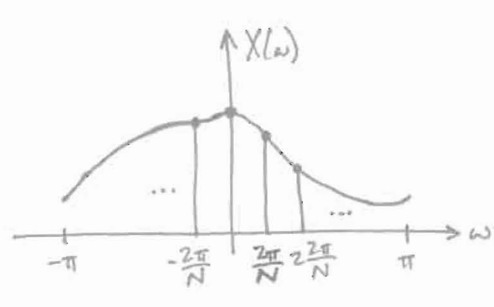
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-ik\frac{2\pi}{N}} \qquad\qquad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{ik\frac{2\pi}{N}n}$$

(a) Express $X_p(k)$ in terms of $X(\omega)$. Provide an illustration of this relationship.

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$X_p[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$= X\left(\frac{2\pi}{N}k\right)$$



For parts (b) – (d), consider the following scheme:

$$x(n) \xrightarrow{DTFT} X(\omega) \xrightarrow{sample\ @\ \Delta\omega = \frac{2\pi}{M}} \tilde{X}(k) \xrightarrow{IDFS} \tilde{x}(n)$$

(b) Suppose that $M = N$. Determine an expression for $\tilde{x}(n)$ in terms of $x(n)$.

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[X\left(\frac{2\pi}{N}k\right)\right] e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\frac{2\pi}{N}k\ell}\right] e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{\ell=-\infty}^{\infty} x[\ell] \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-\ell)}$$

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}k(n-\ell)} = \begin{cases} 1, & n = \ell + rN \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

(c) Determine the condition on the sampling interval $\Delta\omega$ that ensures that $x(n)$ can be perfectly recovered from $\widetilde{X}(k)$ without "time-aliasing".

$\tilde{x}[n]$ consists of copies of $x[n]$ every M samples

Need $M \geq N$

$$\Rightarrow \Delta\omega = \frac{2\pi}{M} \leq \frac{2\pi}{N}$$

(d) Express $X(\omega)$ in terms of $\widetilde{X}(k)$.

$$X(\omega) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi}{N}kn} \right) e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \sum_{n=0}^{N-1} e^{-j(\omega - \frac{2\pi}{N}k)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \frac{1 - e^{-j(\omega - \frac{2\pi}{N}k)N}}{1 - e^{-j(\omega - \frac{2\pi}{N}k)}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] \frac{\sin \frac{1}{2}(\omega - \frac{2\pi}{N}k)N}{\sin \frac{1}{2}(\omega - \frac{2\pi}{N}k)} e^{-j\frac{N-1}{2}(\omega - \frac{2\pi}{N}k)}$$