Solutions to Exam 1

Last name	First name	SID

- You have 1 hour and 45 minutes to complete this exam.

- The exam is closed-book and closed-notes; calculators, computing and communication devices are *not* permitted.

- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

- However, one handwritten and *not photocopied* double-sided sheet of notes is allowed.

- Additionally, you receive Tables 3.1, 3.2, 4.1, 4.2, 5.1, 5.2 from the class textbook.

- If we can't read it, we can't grade it.

- We can only give partial credit if you write out your derivations and reasoning in detail.

- You may use the back of the pages of the exam if you need more space.

*** Good Luck! ***

Problem	Points earned	out of
Problem 1		26
Problem 2		20
Problem 3		22
Problem 4		29
Problem 5		20
Total		117

Problem 1 *(Properties of Systems.)* 26 Points

(a) (4 Pts) True or false: A BIBO stable, causal, linear system must be time-invariant. Give a proof if you believe it's true. Give a counterexample if you believe it's false.

The correct answer is **false**. Some possible counterexamples include $y(t) = \sin(t)x(t)$ and $y(t) = x\left(\frac{t}{2}\right)u(t)$

Note: Many students provided counterexamples that either were not causal (such as $y(t) = x(2t)$), or not BIBO stable (such as $y(t) = tx(t)$).

(b) (6 Pts) True or false: If the (discrete-time) input to a linear, time-invariant system is periodic with period N, then its output must also be periodic with period N. Give a proof if you believe it's true. Give a counterexample if you believe it's false.

The correct answer is **true**. This can be proven by writing out the convolution sum for the LTI system, and then using the periodicity of the input signal.

$$
\begin{align}
y(n + N) &= \sum_{k=-\infty}^{\infty} h(k)x(n + N - k) \tag{1} \\
&= \sum_{k=-\infty}^{\infty} h(k)x(n - k) \tag{2} \\
&= y(n) \tag{3}
\end{align}
$$

Because $y(n + N) = y(n)$ for any n, the output is periodic with period N

(c) (16 Pts) A system with input $x(t)$ and output $y(t)$ can be described as follows:

$$y(t) = \left(\int_{-\infty}^{0} e^{\tau_1} x(t - \tau_1) d\tau_1 \right)^{\left(\int_{-\infty}^{\infty} e^{-|\tau_2 - t|} x(\tau_2) d\tau_2 \right)} \tag{4}$$

Is this a BIBO stable system? Carefully develop your argument.

The correct answer is that this system is *not* BIBO stable. An example input is

$$x(t) = \begin{cases} \frac{e}{1-e}, & -1 \leq t < 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For this input, let's consider $y(t = 0)$: For the exponent, we find

$$\int_{-\infty}^{\infty} e^{-|\tau_2 - t|} x(\tau_2) d\tau_2 = \frac{e}{1 - e} \int_{-1}^{0} e^{-|\tau_2|} d\tau_2 = \frac{e}{1 - e} \int_{0}^{1} e^{-\tau_2} d\tau_2 \tag{6}$$

$$= \frac{e}{1 - e} \left(1 - e^{-1} \right) = -1 \tag{7}$$

Hence, the exponent is -1. It is trivial to see that the base is zero, which leads to an unbounded output ("1/0").

Credit was also given for a correct argument showing that both the base and the exponent are bounded if the input signal $x(t)$ is bounded. Note that this does *not* imply stability: If the exponent is negative, and the base very small, the resulting output is very large (see the counter-example above). However, the argument of showing that both the base and the exponent are bounded for bounded input $x(t)$ is sufficient to show that a close relative of the above system, namely the system

$$y(t) = \left(\int_{-\infty}^{0} e^{\tau_1} x(t - \tau_1) d\tau_1 \right)^{\left| \int_{-\infty}^{\infty} e^{-|\tau_2 - t|} x(\tau_2) d\tau_2 \right|}. \tag{8}$$

is (BIBO) stable.

Problem 2 *(Convolution.)* 20 Points

Let

$$x(t) = \begin{cases} -t, & -1 \le t < 0, \\ 1, & 0 \le t \le 1, \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

and

$$y(t) = \begin{cases} 1, & |t| \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

Sketch $x(t) * y(t)$, carefully labeling both axes. *Note:* The correct plot with the correct labeling on both axes will get full credit; there is no requirement to write down formulas.

We can compute the convolution of $x(t)$ and $y(t)$ directly, or use linearity to compute two simpler convolutions, $x(t) * y(t) = (x_1(t) + x_2(t)) * y(t) = x_1(t) * y(t) + x_2(t) * y(t)$.
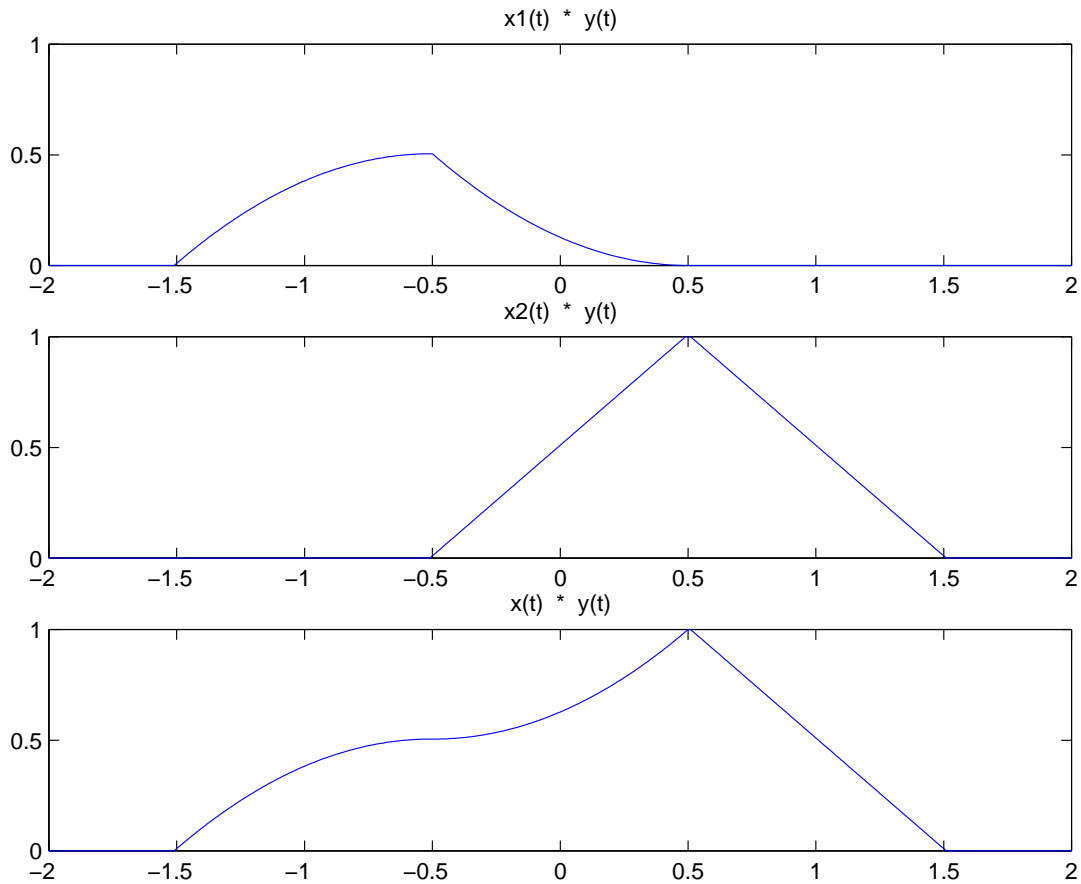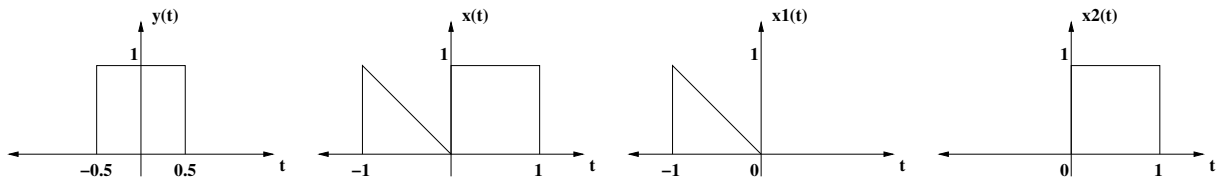
Problem 3 *(Short-time Fourier Transform.)*   22 Points

To compute a Fourier transform requires us to know the entire signal $x(t)$, for $-\infty < t < \infty$. In practice, when doing spectral analysis, we cannot usually wait that long. One way out of this dilemma is to perform what people often call the *short-time Fourier transform*: Instead of looking at the actual Fourier transform of $x(t)$, we consider

$$x_w(t) = x(t)w(t), \tag{11}$$

where $w(t)$ is sometimes called a "window". The short-time Fourier transform of $x(t)$ with respect to the window $w(t)$ is then simply $X_w(j\omega)$, the Fourier transform of $x_w(t)$.

(a) (16 Pts) An obvious first choice of a window is simply to cut the signal $x(t)$ at some point, i.e.,

$$w(t) = \begin{cases} 1, & |t| \leq T \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

For the special case $x(t) = \cos(\omega_0 t)$ find the formula for the short-term Fourier transform $X_w(j\omega)$. Give a sketch of $X_w(j\omega)$. Use $T = 5\pi/\omega_0$ for the sketch.

In OWN Table 4.2 of Fourier transform pairs, we can find the Fourier transforms of $x(t)$ and $w(t)$.

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$
$$W(j\omega) = \frac{2\sin(\omega T)}{\omega}$$

The multiplication property of the Fourier transform, given in OWN Table 4.1, tells us that multiplication in the time domain corresponds to convolution in the frequency domain. Remember that convolution of a function with an impulse simply shifts the function.

$$\begin{aligned} X_w(j\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) W(j(\omega - \theta)) d\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \pi(\delta(\theta - \omega_0) + \delta(\theta + \omega_0)) \frac{2\sin((\omega - \theta)T)}{\omega - \theta} d\theta \\ &= \frac{\sin((\omega - \omega_0)T)}{\omega - \omega_0} + \frac{\sin((\omega + \omega_0)T)}{\omega + \omega_0} \end{aligned}$$

Thus the Fourier transform of the windowed signal $x_w(t)$ is the sum of two sinc functions shifted to center around $\omega = \pm\omega_0$, each with peak height T. A plot of $X_w(j\omega)$ is shown for $T = 1$ and $\omega_0 = \frac{5\pi}{T}$.

$X_\omega(j\omega)$
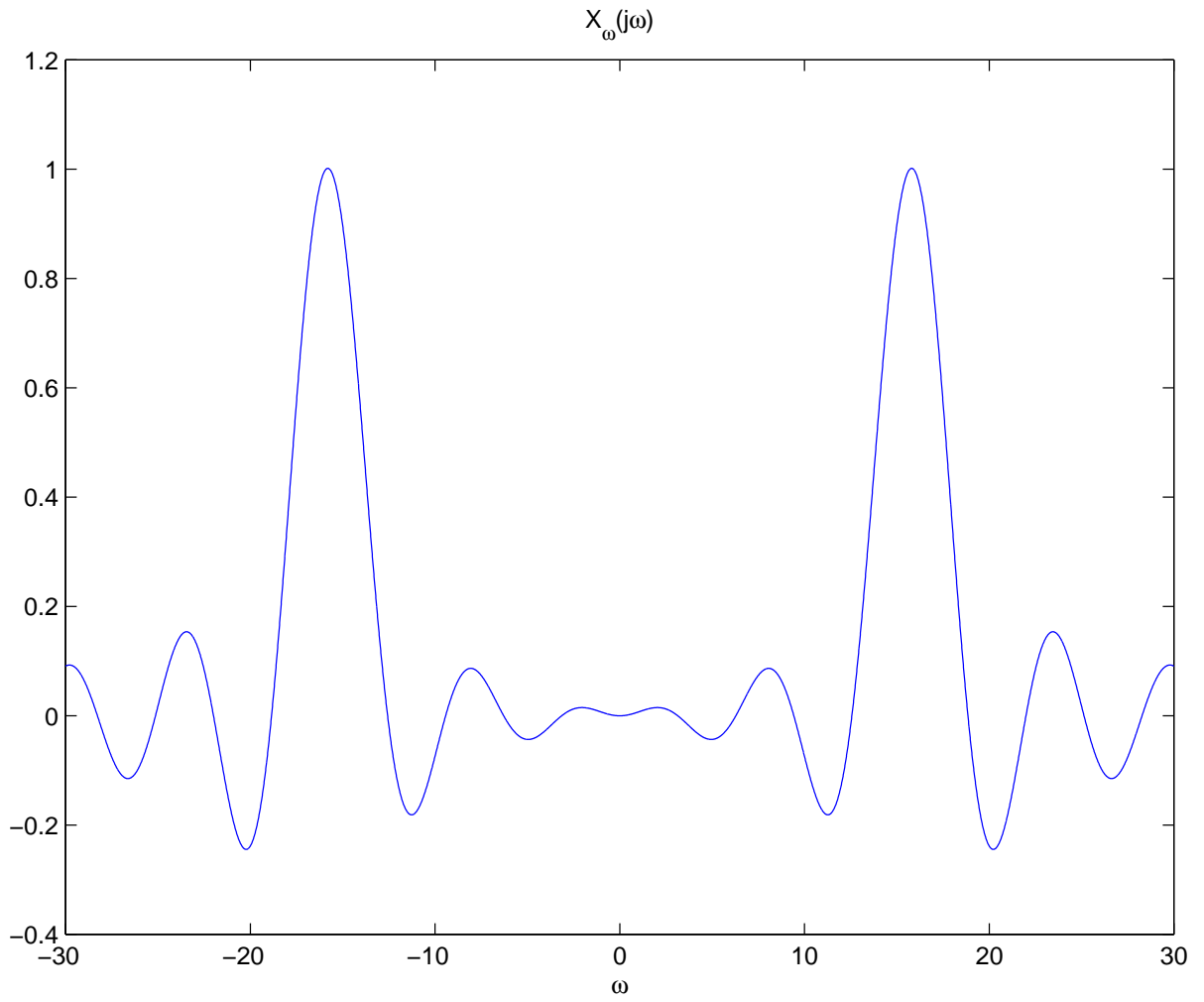
(b) (6 Pts) Explain what happens as T and ω_0 are varied. You may use sketches to do this. *Hint: Consider the "extreme" cases, T becomes very large or very small.*

As T becomes large and ω_0 is held constant, the sinc functions become narrow and tall. In the limit as $T \to \infty$, the sinc functions get squeezed into impulses centered at $\pm\omega_0$, and $X_w(j\omega) \to X(j\omega)$. This makes sense because as the time window $w(t)$ becomes infinitely wide, windowing will have no effect.

On the other hand, as T becomes small and ω_0 is held constant, the sinc functions become wide and short. In the limit as $T \to 0$, the sinc functions flatten, and $X_w(j\omega) \to 0$. This also makes sense because as the window becomes infinitely narrow, it does not pass the signal $x(t)$.

Now varying ω_0 while holding T constant has the effect of shifting the two sinc functions outwards for large values of ω_0, and inwards for small values of ω_0.
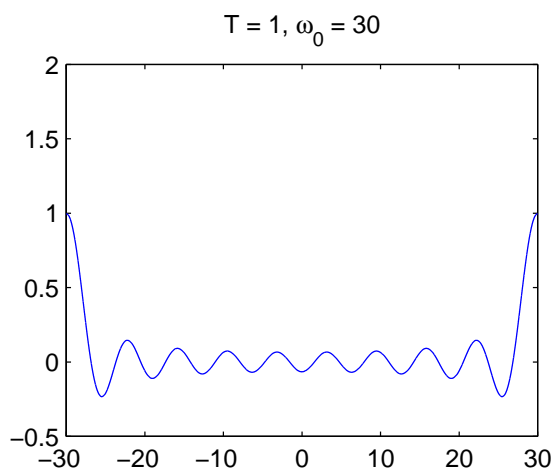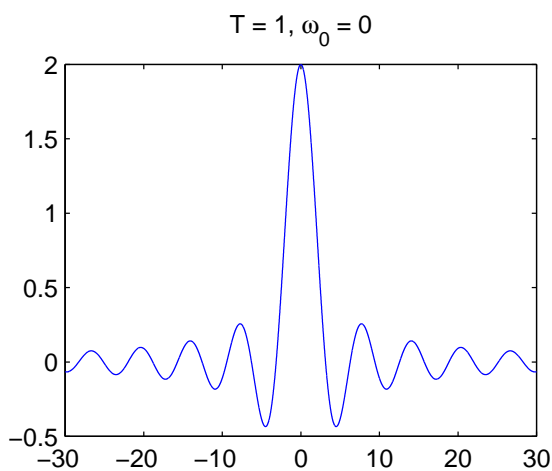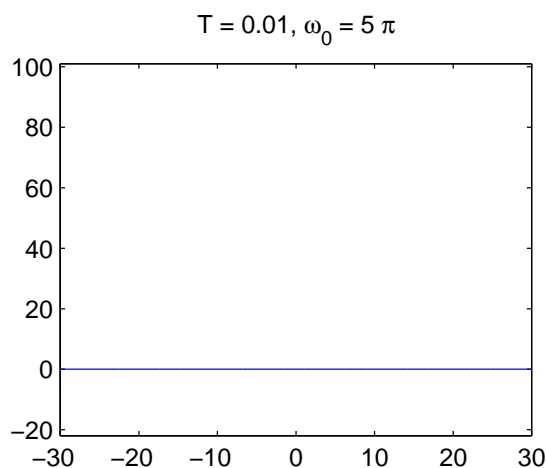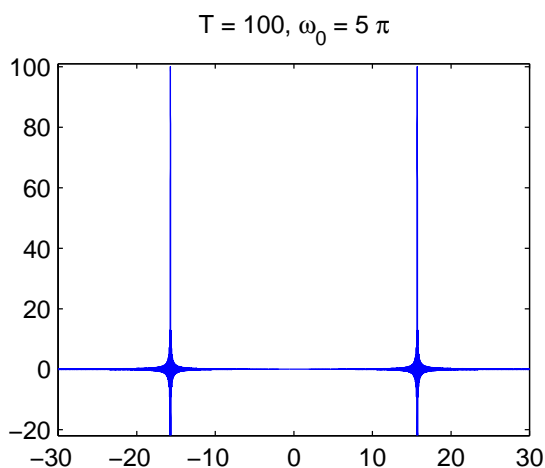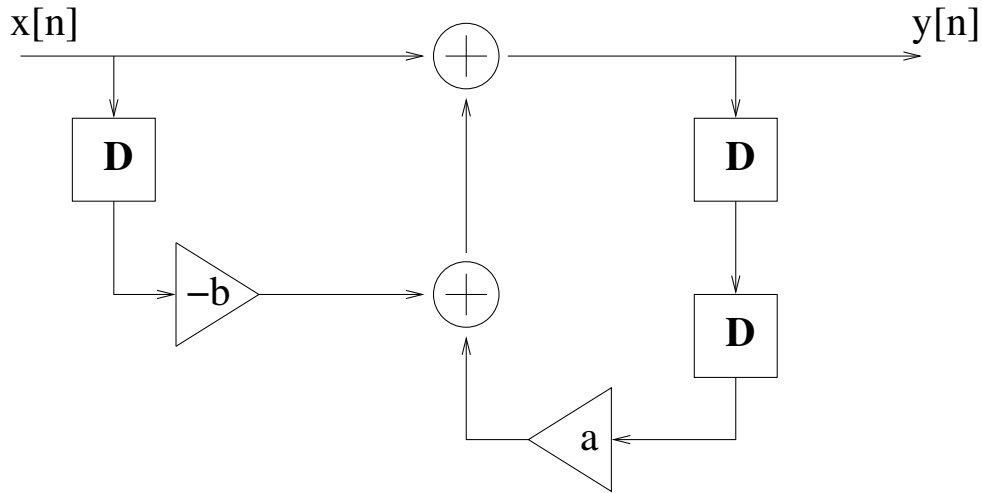
Problem 4 *(System.)* 29 Points

Consider the causal, linear, time-invariant discrete-time system that obeys the following difference equation:

$$y[n] - ay[n-2] \quad = \quad x[n] - bx[n-1] \tag{13}$$

(a) (6 Pts) Give a system diagram, using only multipliers, adders, and delay elements.

(b) (7 Pts) Find the frequency response of this system.

We take the DTFT of both sides of the difference equation to see that

$$Y(e^{j\omega}) - ae^{-j\omega 2}Y(e^{j\omega}) = X(e^{j\omega}) - be^{-j\omega}X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 - be^{-j\omega}}{1 - ae^{-j\omega 2}}$$

(c) (8 Pts) Find the impulse response of this system.

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega 2}} - \frac{be^{-j\omega}}{1 - ae^{-j\omega 2}}$$

One way to find the impulse response is to combine the entry in Table 5.2 which shows that the inverse transform of $Z(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$ is $z[n] = a^n u[n]$ with the time expansion property in Table 5.1

This combination shows that the inverse transform of $\frac{1}{1 - ae^{-j\omega 2}}$ is given by

$$z_{(2)}[n] = \begin{cases} a^{n/2}u[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Finally, using the linearity of the DTFT, and applying the time shifting property in Table 5.1 to the second term in $H(e^{j\omega})$, we see that

$$h[n] = \begin{cases} a^{n/2}u[n/2] & n \text{ even} \\ -ba^{(n-1)/2}u[(n-1)/2] & (n-1) \text{ even} \end{cases}$$

$$h[n] = \begin{cases} a^{n/2}u[n] & n \text{ even} \\ -ba^{(n-1)/2}u[n] & n \text{ odd} \end{cases}$$

A second way to find the impulse response is by computing the partial fraction expansion of $H(e^{j\omega})$

$$H(e^{j\omega}) = \frac{1 - be^{-j\omega}}{1 - ae^{-j\omega 2}} = \frac{A}{1 - \sqrt{a}e^{-j\omega}} + \frac{B}{1 + \sqrt{a}e^{-j\omega}}$$

$$1 - be^{-j\omega} = A + A\sqrt{a}e^{-j\omega} + B - B\sqrt{a}e^{-j\omega}$$

Now, equating the constant terms and the $e^{-j\omega}$ terms, we see that

$$1 = A + B$$
$$-b = \sqrt{a}A - \sqrt{a}B$$

We can solve this system of equations to find that $A = \frac{\sqrt{a}-b}{2\sqrt{a}}$ and $B = \frac{\sqrt{a}+b}{2\sqrt{a}}$

$$H(e^{j\omega}) = \frac{\frac{\sqrt{a}-b}{2\sqrt{a}}}{1 - \sqrt{a}e^{-j\omega}} + \frac{\frac{\sqrt{a}+b}{2\sqrt{a}}}{1 + \sqrt{a}e^{-j\omega}}$$

Finally, we use Table 5.2 to find

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$$h[n] = \frac{\sqrt{a}-b}{2\sqrt{a}}(\sqrt{a})^n u[n] + \frac{\sqrt{a}+b}{2\sqrt{a}}(-\sqrt{a})^n u[n] \tag{14}$$

$$= \begin{cases} a^{n/2}u[n] & n \ even \\ -ba^{(n-1)/2}u[n] & n \ odd \end{cases} \tag{15}$$

(d) (8 Pts) For what values of a and b is the system stable?

A system is stable if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

For the impulse response in part *(c)*

$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{\substack{k=0, \\ even}}^{\infty} |a|^{k/2} + \sum_{\substack{k=1, \\ odd}}^{\infty} |b||a|^{(k-1)/2} \tag{16}$$

$$= \sum_{m=0}^{\infty} |a|^m + \sum_{m=0}^{\infty} |b||a|^m \tag{17}$$

These two summations are finite, and therefore the system is stable, when $|a| < 1$ and $|b| < \infty$

Problem 5 *(Non-linear System.)* 20 Points

A simple non-linear system is characterized by

$$y(t) = \left(\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \right)^2, \quad \text{where} \quad h(t) = 2\mathrm{sinc}(2t) \tag{18}$$

We would like to approximate this system by a linear and time-invariant system. The following two systems have been suggested. The first suggested LTI system has impulse response

$$h_1(t) = h(t). \tag{19}$$

and we denote its output by $\tilde{y}_1(t)$. The second suggested LTI system has frequency response

$$H_2(j\omega) = \begin{cases} 2 - |\omega|/(2\pi), & |\omega| \leq 2\pi \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

and we denote its output by $\tilde{y}_2(t)$.

(a) (15 Pts) For the special input signal $x(t) = 3\mathrm{sinc}(3t)$, find the errors

$$E_m = \int_{-\infty}^{\infty} (y(t) - \tilde{y}_m(t))^2 dt, \quad \text{for } m = 1, 2. \tag{21}$$

Which system incurs a smaller error, $h_1(t)$ or $h_2(t)$?

The first task is to convert all signals to the frequency domain. Here, this is easy, simply using the tables:

$$H(j\omega) = H_1(j\omega) = \begin{cases} 1, & |\omega| \leq 2\pi \\ 0, & \text{otherwise}, \end{cases} \tag{22}$$

and

$$X(j\omega) = \begin{cases} 1, & |\omega| \leq 3\pi \\ 0, & \text{otherwise}. \end{cases} \tag{23}$$

Next, we define the signal

$$z(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau, \quad \text{where} \quad h(t) = 2\mathrm{sinc}(2t), \tag{24}$$

or, in frequency domain,

$$Z(j\omega) = H(j\omega)X(j\omega) = H(j\omega) \quad \text{in this especially simple case.} \tag{25}$$

In terms of $z(t)$, we can simply write $y(t) = z^2(t)$, and using the multiplication property,

$$Y(j\omega) = \frac{1}{2\pi}Z(j\omega) * Z(j\omega) = \frac{1}{2\pi}H(j\omega) * H(j\omega). \tag{26}$$

Since $H(j\omega)$ is just a box, let's do a graphical ("flip-and-drag") convolution. A box convolved with a box gives a triangle, and we only need to know three things. Straightforwardly, we find that it's a triangle from -4π to 4π, of height 2 at $\omega = 0$, as illustrated in the figure.

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Next, we find the outputs of two proposed LTI systems when the input is the given (very simple) $x(t)$. We find
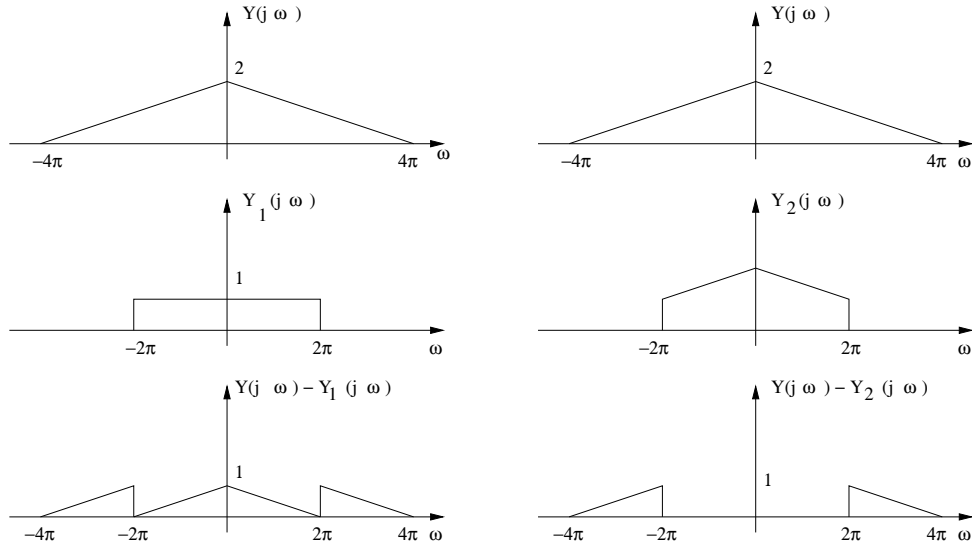
$$\tilde{Y}_1(j\omega) = H_1(j\omega)X(j\omega) = H(j\omega) \quad \text{and} \quad \tilde{Y}_2(j\omega) = H_2(j\omega)X(j\omega) = H_2(j\omega). \qquad (27)$$

To find the errors, we use Parseval:

$$E_m = \int_{-\infty}^{\infty} (y(t) - \tilde{y}_m(t))^2 dt, \text{ for } m = 1, 2. \qquad (28)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega) - \tilde{Y}_m(j\omega)|^2 d\omega \qquad (29)$$

The terms $Y(j\omega) - \tilde{Y}_m(j\omega)$ can be read straight out of the graphs.

Now, we still have to *square* the error spectra, and then integrate. One way of doing this quickly and efficiently is to note that the error for the first system consists of 4 little triangles (that need to be squared). Hence, it is sufficient to just calculate it for one single such triangle, i.e.,

$$E_1 = 4 \cdot \frac{1}{2\pi} \int_0^{2\pi} (1 - \omega/(2\pi))^2 d\omega = 4 \cdot \frac{1}{2\pi} \int_0^{2\pi} (\tilde{\omega}/(2\pi))^2 d\tilde{\omega} \qquad (30)$$

$$= 4 \cdot \frac{1}{2\pi} \frac{1}{(2\pi)^2} \left[\frac{\omega^3}{3} \right]_0^{2\pi} = \frac{4}{3}, \qquad (31)$$

using the substitution $\tilde{\omega} = 2\pi - \omega$. The second error looks very similar, but it only has *two*, rather than four, little triangles, hence

$$E_2 = \frac{E_1}{2} = \frac{2}{3}. \qquad (32)$$

Clearly, the system H_2 incurs a smaller error.

(b) (5 Pts) Repeat Part (a) for the input signal $x(t) = \cos(\frac{2\pi}{3}t)$, this time finding the *power* of the error signal $y(t) - \tilde{y}_m(t)$, for $m = 1, 2$. In this case, which system incurs a smaller error, $h_1(t)$ or $h_2(t)$?

The first step is again to notice that the filter $h(t)$ leaves the input signal unchanged, i.e., the desired output signal is merely

$$y(t) = \cos^2(\frac{2\pi}{3}t). \tag{33}$$

You can again use the multiplication property, or you can do it directly to find

$$y(t) = \frac{1}{2} + \frac{1}{2}\cos(\frac{4\pi}{3}t). \tag{34}$$

The output of the first candidate system is very easy to find: as we said, the system $h(t)$ leaves the input unchanged, hence:

$$\tilde{y}_1(t) = \cos(\frac{2\pi}{3}t). \tag{35}$$

The power of the first error P_1 is

$$P_1 = \lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T}(y(t) - \tilde{y}_1(t))^2 dt = \lim_{T \to \infty} \frac{1}{2T}\int_{-T}^{T}\left(1 + \frac{1}{2}\cos(\frac{4\pi}{3}t) - \cos(\frac{2\pi}{3}t)\right)^2 dt \tag{36}$$

The signal in parentheses is a periodic signal of fundamental period 3, hence

$$P_1 = \frac{1}{3}\int_{0}^{3}\left(1 + \frac{1}{2}\cos(\frac{4\pi}{3}t) - \cos(\frac{2\pi}{3}t)\right)^2 dt \tag{37}$$

I would solve this using Parseval: the Fourier coefficients can be read out directly: $a_0 = \frac{1}{2}$, $a_1 = a_{-1} = -\frac{1}{2}$, and $a_2 = a_{-2} = \frac{1}{4}$. All other Fourier coefficients are zero. Hence,

$$P_1 = a_0^2 + 2a_1^2 + 2a_2^2 = \frac{1}{4} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{16} = \frac{7}{8}. \tag{38}$$

For the second candidate system, we can use the frequency response property, rewriting $\cos(\frac{2\pi}{3}t) = \frac{1}{2}(e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t})$ as follows:

$$\tilde{y}_2(t) = \frac{1}{2}\left(H_2(j\frac{2\pi}{3})e^{j\frac{2\pi}{3}t} + H_2(-j\frac{2\pi}{3})e^{-j\frac{2\pi}{3}t}\right) \tag{39}$$

But thanks to the nice symmetry of H_2, we find $H_2(j\frac{2\pi}{3}) = H_2(-j\frac{2\pi}{3}) = \frac{5}{3}$, and hence

$$\tilde{y}_2(t) = \frac{5}{3}\cos(\frac{2\pi}{3}t). \tag{40}$$

By the same token, the error signal $y(t) - \tilde{y}_2(t)$ is periodic with period 3, but the Fourier coefficients are now $a_0 = \frac{1}{2}$, $a_1 = a_{-1} = -\frac{5}{6}$, and $a_2 = a_{-2} = \frac{1}{4}$, and therefore

$$P_1 = a_0^2 + 2a_1^2 + 2a_2^2 = \frac{1}{4} + 2 \cdot \frac{25}{36} + 2 \cdot \frac{1}{16} \approx 1.764. \tag{41}$$

Here, this time, system H_1 incurs a smaller error.

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