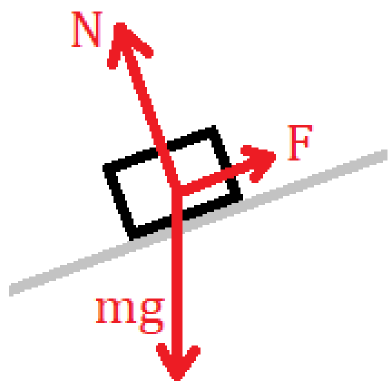
1 Yildiz 2019 Midterm 2 Problem 1 Solution - Icy Ramp



First, let's find the friction needed for a block at rest to stay at rest. Gravity, friction, and the normal force all act on the block, as shown on the FBD. Since a block would slide down the ramp without friction, we'll guess that the frictional force points upslope.

From force balance in the normal direction, we find that

$$N = mg\cos(\theta)$$

and

$$F = mg\sin(\theta).$$

We also know that $F \leq N\mu_s$, so $mg\sin(\theta) \leq mg\cos(\theta)\mu(x)$, giving us the condition that

$$\mu(x) = Ax > \tan(\theta).$$

This means that a block must reach at least a height $x_{min} = \tan(\theta)/A$ for friction to keep it from sliding back down.

The question now is how fast the block should be moving to reach x_{min}. We can find this with conservation of energy. When sliding, the frictional force backwards is just $F = N\mu = -Amg\cos(\theta)x$, which is negative because it's pointing down the slope. The total work done by friction while the block slides a distance x_{min} is then

$$W = \int F dx = \int_0^{x_{min}} -Amg\cos(\theta)xdx = -Amg\cos(\theta)\frac{x_{min}^2}{2}.$$

From the start of the block's slide to its end at x_{min}, the net change in total energy will be the negative work done by friction. We can write this as

$$\Delta E = U_f - K_i = mgx_{min}\sin(\theta) - \frac{mv_0^2}{2} = -Amg\cos(\theta)\frac{x_{min}^2}{2} = W.$$

We can now just solve for v_0 and plug in the fact that $x_{min} = \tan(\theta)/A$. Once simplified, this gives

$$v_0 = \sqrt{\frac{3g\sin^2(\theta)}{A\cos(\theta)}}.$$

Checking a few limits, we see that when $\theta = 0$ the min velocity is zero, and when $\theta = \pi/2$ - when the ramp is vertical - no velocity is large enough, which makes sense.

2 Yildiz 2019 Midterm 2 Problem 2 Solution - Rocket

The key equation here is $F_{ext} = -\frac{dm}{dt}v_{rel} + m(t)\frac{dv}{dt}$. There's no external force, so $\frac{dm}{dt}v_{rel} = m(t)\frac{dv}{dt}$; in words, this says that the only force on the mass m is the recoil force from ejecting the fuel. Since the problem's only asking about the speed, not the velocity, and since it's clear in which direction a rocket should move, we won't worry about the sign of v. Let's use the shorthand $K = \frac{dm}{dt}$.

(a) From the above equation, at $t = 0$, $a = \frac{dv}{dt} = K\frac{v_{rel}}{m_0}$. The problem tells us that $K = \frac{m_0}{120\text{s}}$, so

$$a = \frac{2400\text{m/s}}{120\text{s}} = 20\text{m/s}.$$

(b) The challenge now is to find how this rocket moves. If you don't have the rocket equation on your formula sheet, here's how it's derived.

We know that the mass left in the rocket as a function of time is given by $m(t) = m_0 - Kt$. This fact buys us the following differential equation, which we can solve:

$$K v_{rel} = (m_0 - Kt)\frac{dv}{dt}$$

$$\frac{K v_{rel}}{m_0 - Kt} dt = dv$$

To integrate, we let t_f be the time at the end of the burn, v_f be the final velocity, and m_f be the mass left in the rocket after the burn.

$$\int_0^{t_f} \frac{K v_{rel}}{m_0 - Kt} dt = \int_0^{v_f} dv$$

$$\Big[- v_{rel} \ln(m_0 - Kt) \Big]_0^{t_f} = v_f$$

$$v_{rel} \ln\left(\frac{m_0}{m_0 - Kt_f}\right) = v_f$$

$$v_f = v_{rel} \ln\left(\frac{m_0}{m_f}\right) = (2400\text{m/s}) \ln(4).$$

This v_f is the solution.

Problem 3

a) If r is constant, $\Delta s = r\Delta\theta$. As $\Delta\theta$ gets infinitesemal, $r(\theta)$ is approximately constant so $ds = r(\theta)d\theta = r_0 d\theta + \beta\theta d\theta$.

b)

$$s = \int_{s=0}^{s} ds = \int_{\theta=0}^{\theta} (r_0 + \beta\theta)\, d\theta = r_0\theta + \frac{\beta}{2}\theta^2$$

c)

$$vt = r_0\theta + \frac{\beta}{2}\theta^2 \qquad \Longrightarrow \qquad \frac{\beta}{2}\theta^2 + r_0\theta - vt = 0$$

$$\Longrightarrow \qquad \theta = \frac{-r_0 \pm \sqrt{r_0^2 + 2\beta vt}}{\beta}$$

Since the rotation of the CD is said to be positive in the problem statement, we take the positive θ, so

$$\theta(t) = \frac{-r_0 + \sqrt{r_0^2 + 2\beta vt}}{\beta}$$

d)

$$\omega(t) = \frac{d\theta}{dt} = \frac{d}{dt}\frac{1}{\beta}\sqrt{r_0^2 + 2\beta vt} = \frac{1}{\beta}\frac{1}{2}\frac{2\beta v}{\sqrt{r_0^2 + 2\beta vt}} = \frac{v}{\sqrt{r_0^2 + 2\beta vt}}$$

e)

$$\alpha(t) = \frac{d\omega}{dt} = \frac{d}{dt}v\left[r_0^2 + 2\beta vt\right]^{-1/2} = \frac{-v(2\beta v)}{2(r_0^2 + 2\beta vt)^{3/2}} = \frac{-\beta v^2}{(r_0^2 + 2\beta vt)^{3/2}}$$

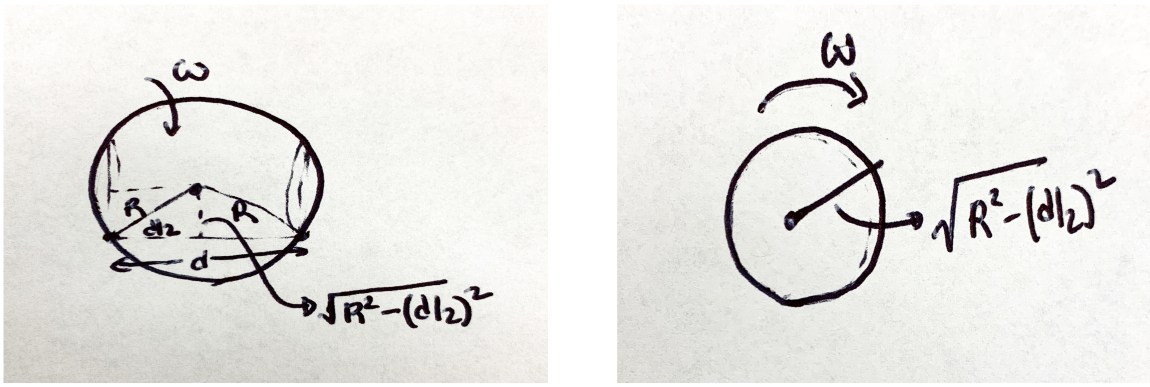
So the angular acceleration α is not constant.

Problem 4

(a) Note that the sphere rolls around the two points-of-contact with the rail (see figure 1). The rolling without slipping condition needs to be written with respect to the radius of this circle:

$$\boxed{v_{cm} = \omega\sqrt{R^2 - (d/2)^2} \equiv \omega R'.} \tag{1}$$

It is going to be important to keep track of R and R' throughout the problem.



(a) Front view

(b) Side view

Figure 1

(b) Draw the free-body diagram for the object. This results in an equation for torques and an equation for forces.

$$ma = mg\sin\theta - 2f_s \tag{2}$$
$$I\alpha = 2f_s\,R', \tag{3}$$

where we are calculating the torque around the axis passing through the center of mass with the CM. We have 2 equations and 3 unknowns, which means we need an extra equation. We know that the sphere is rolling without slipping, which implies

$$\alpha = \frac{a}{R'}. \tag{4}$$

You could get this either by differentiating part (a), or by noting that the lever arm of the torque is R'. We now solve for a using $I_{cm} = 2/5MR^2$ (not R'^2)

$$\boxed{a = \frac{g\sin\theta}{\left(\frac{I}{mR'^2} + 1\right)} = \frac{g\sin\theta}{\frac{2}{5}\frac{R^2}{R'^2} + 1},} \tag{5}$$

where $R' = \sqrt{R^2 + (d/2)^2}$.

(c) Use conservation of energy

$$mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}^2 = \frac{1}{2}I\frac{v_{cm}^2}{R'^2} + \frac{1}{2}mv_{cm}^2. \tag{6}$$

Solve for v_{cm} to obtain:

$$\boxed{v_{cm} = \sqrt{\frac{gh}{\frac{1}{5}\frac{R^2}{R'^2} + \frac{1}{2}}}.} \tag{7}$$

Problem 5

a) We want a function $\rho(r)$ of r that decreases linearly from 0 to R, with the boundary conditions that

$$\rho(r=0) = \rho_0 \tag{1}$$

and

$$\rho(r=R) = \frac{\rho_0}{4} \tag{2}$$

let $\rho(r) = \rho_0(1 - kr)$. Applying the second boundary condition we find $k = \frac{3}{4R}$
and

$$\boxed{\rho(r) = \rho_0(1 - \frac{3}{4R}r)} \tag{3}$$

b) The total mass is given by

$$M_{tot} = \int dm \tag{4}$$

using the relationship between density and mass, $dm = \rho dV$. This gives

$$M_{tot} = \int_0^R \rho(r) dV \tag{5}$$

and

$$M_{tot} = \int_0^R \int_0^\pi \int_0^{2\pi} \rho_0(1 - \frac{3}{4R}r) r^2 \text{Sin}(\theta) dr d\theta d\phi \tag{6}$$

the angular integrals yield

$$M_{tot} = 4\pi \int_0^R \rho_0(1 - \frac{3}{4R}r) r^2 dr \tag{7}$$

and the radial integral gives

$$M_{tot} = \pi\rho_0 \left[\frac{4R^3}{3} - \frac{3R^3}{4}\right] \tag{8}$$

Simplifying, we arrive at our final answer

$$\boxed{M_{tot} = \frac{7\pi\rho_0 R^3}{12}} \tag{9}$$

c) The force of gravity acts like all the mass of the planet is concentrated at the center. Newtonian gravitational force is given by

$$F_g = G\frac{mm_2}{r^2} = m_p g \tag{10}$$

divide both sides by m to get

$$g = G\frac{m_p}{r^2} \tag{11}$$

use the answer in the previous part to get

$$\boxed{g = G\frac{7\pi\rho_0 R^3}{12r^2}} \tag{12}$$