EECS 16A Designing Information Devices and Systems I
Spring 2020

Final Exam Instructions

Read the following instructions before the exam.

Format & How to Submit Answers

There are 16 problems (4 introductory questions, and 12 exam questions, comprising 50 subparts total) of varying numbers of points. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can. Don't get bogged down in calculations; if you are having trouble with one problem, there may be easier points available later in the exam!

All answers will be submitted to the Gradescope "Final Exam" Assignment (`https://www.gradescope.com/courses/83747/assignments/500716`). All subparts, except introductory questions, are multiple choice and are **worth 3 points each**. **There are 145 points possible on the exam, but your final score will be taken out of 100 points**. This means that a score of 75/145, normally 51.7%, will be bumped up to 75/100, or 75%. You cannot score more than 100% on this exam.

Partial credit may be given for certain incorrect answer choices for some problems. There is no penalty for incorrect answers.

Post any content or clarifying questions privately on Piazza. There will be no exam clarifications; if we find a bug on the exam, that sub-question will be omitted from grading.

Timing & Penalties

You have 180 minutes for the exam, with a 5 minute grace period. After the 5 minute grace period ends, exam scores will be penalized exponentially as follows: an exam that is submitted N minutes after the end of the grace period will lose 2^N points. **The exam will become available at your personalized link at 8:10 am PT; the grace period will expire at 11:15 am PT.** If your submission is timestamped at 11:16 am PT, you will lose 2 points; if it is timestamped at 11:18 am PT, you will lose 8 points.

We will count the latest time at which you submit any question as your exam timestamp. Do NOT edit or resubmit your answers after the deadline. We recommend having all of your answers input and submitted by 11:10 am; it is your responsibility to submit the exam on time.

If you cannot access your exam at your link by 8:15 am, please email eecs16a@berkeley.edu. If you are having technical difficulties submitting your exam, you can email your answers (either typed or scanned) to eecs16a@berkeley.edu.

Academic Honesty

This is an open-note, open-book, open-internet, and **closed-neighbor** exam. You may use any calculator or calculation software that you wish, including Wolfram-Alpha and Mathematica. **No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.**

We have zero tolerance against violation of the Berkeley Honor Code. Given supporting evidence of cheating, we reserve the right to automatically fail all students involved and report the instance to the student conduct committee. Feel free to report suspicious activity through this form. (`https://forms.gle/akhBsHVr1WG29Ufg9`).

Our advice to you: if you can't solve a particular problem, move on to another, or state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution.

Good luck!

EECS 16A Designing Information Devices and Systems I
Spring 2020 Final Exam

1. **Pledge of Academic Integrity (2 points)**
   By my honor, I affirm that:

   (1) this document, which I will produce for the evaluation of my performance, will reflect my original, bona fide work;

   (2) as a member of the UC Berkeley community, I have acted and will act with honesty, integrity, and respect for others;

   (3) I have not violated—nor aided or abetted anyone else to violate—nor will I—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

   (4) I have not committed, nor will I commit, any act that violates—nor aided or abetted anyone else to violate—the UC Berkeley Code of Student Conduct.

   Write your name and the current date as an acknowledgement of the above. (See Gradescope)

2. **Administrivia (1 point)**
   I know that I will lose 2^n points for every n minutes I submit after the exam submission grace period is over.

   For example, if the exam becomes available at my personalized link at 8:10 a.m. PT; the grace period will expire at 11:15 a.m. PT. If my submission is timestamped at 11:16 a.m. PT, I will lose 2 points; if it is timestamped at 11:18 a.m. PT, I will lose 8 points.

   ○ Yes

3. **What are you looking forward to this summer? (2 points)**

4. **Tell us about something that makes you happy. (2 points)**

5. Matrix Properties (9 points)

What can you say about the following matrices? For each matrix that is given, '$*$' denotes a nonzero entry, while '0' denotes an entry equal to zero.

(a) Let the following matrix

$$A = \left[\begin{array}{ccc|c} * & 0 & * & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & * \end{array}\right]$$

be an augmented matrix that corresponds to a system of 4 linear equations in 3 unknowns. How many solutions does this system of equations have?

(A) One solution

(B) Not enough information to determine

(C) No solutions

(D) Infinitely many solutions

Solution: (C), No solutions.
The second row of matrix A gives $x_3 = 0$, while the fourth one gives $x_3 \neq 0$, hence the system of equations is inconsistent.

(b) Let the following matrix

$$B = \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{array}\right]$$

be an augmented matrix that corresponds to a system of 4 linear equations in 3 unknowns. Select all that apply.

(A) The 4×3 matrix corresponding to the original system of equations has a non-trivial nullspace

(B) B is in reduced row echelon form

(C) B is in row echelon form

(D) B corresponds to a consistent set of linear equations

Solution: (C), The matrix is in row echelon form. The matrix cannot be in reduced row echelon form as the pivots are not known to be 1's, and the entries above the pivots are not zero. The system of linear equations is not consistent because of the last row which says $0 = *$. The matrix to the left of the separator has linearly independent columns and therefore can only have $\{\begin{bmatrix}0 & 0 & 0\end{bmatrix}^T\}$ as the nullspace.

(c) Let the following matrix

$$C = \left[\begin{array}{cccc|c} * & * & 0 & * & * \\ 0 & * & * & * & 0 \\ 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

be an augmented matrix that corresponds to a system of 4 linear equations in 4 unknowns. Which statements are guaranteed to be true? Select all that apply.

(A) The system has exactly one basic variable

(B) The system has exactly two free variables

(C) The system has exactly one free variable

(D) The system has exactly two basic variables

(E) The system of equations is consistent

Solution: (E), This system is consistent.
Since both elements c_{25}, c_{35} are zero the matrix can be brought in rref without having a line that will be equal to $[0, 0, 0, ...|1]$. Hence, the system will be consistent. We cannot, however, deduce anything about the number of free and basic variables, since that depends on whether rows 2 and 3 are linearly independent.

6. Splotchy Writing v2.0 (9 points)

It doesn't matter whether Professor Courtade writes in a sharpie or on an iPad, he still has terrible handwriting. The following is a (hypothetical) passage from lecture notes, and the smudges are labeled (1), (2), ..., (10). Your task is to identify correct expressions for some of the smudges.

The least squares solution derived in class was under the assumption that the norm $\|\cdot\|$ was the Euclidean one. However, if we are given an arbitrary inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$, we can still solve the least squares problem

$$\min_{\vec{x}\in\mathbb{R}^m} \|A\vec{x}-\vec{b}\|, \tag{1}$$

where $A \in \mathbb{R}^{(1)\times(2)}$ and $\vec{b} \in \mathbb{R}^{(3)}$ are given, and $\|\cdot\|$ is the norm on $\mathbb{R}^n$ induced by the inner product $\langle\cdot,\cdot\rangle$. The key difference between this and the Euclidean setting is that we must introduce the so-called adjoint of A, denoted by A^*, which is the unique (4) $\times$ (5) matrix satisfying

$$(A^*\vec{y})^T\vec{x} = \langle\vec{y}, A\vec{x}\rangle$$

for all choices of vectors $\vec{x} \in \mathbb{R}^{(6)}$ and $\vec{y} \in \mathbb{R}^{(7)}$. In analogy to the Euclidean case, the solutions of the (non-Euclidean) least squares problem (1) are precisely those solutions to the system of "normal" equations

$$A^*A\vec{x} = A^*\vec{b}.$$

Using the fact that $N(A^*A) = N(A)$ (these are both subspaces of $\mathbb{R}^{(8)}$), we conclude that if $\text{rank}(A) =$ (9), then A^*A is invertible, and $\vec{x} = (A^*A)^{-1}A^*\vec{b} \in \mathbb{R}^{(10)}$ is the unique solution to the (non-Euclidean) least squares problem (1).

What is the correct value of each smudge below?

(a) Smudge (3)

(A) not enough information to determine

(B) m

(C) n

(b) Smudge (6)

(A) n

(B) not enough information to determine

(C) m

(c) Smudge (8)

(A) m

(B) n

(C) not enough information to determine

Solution: Answers: (C), (C), (A).

1 n: Since $\|\cdot\|$ is a norm on $\mathbb{R}^n$, $A\vec{x}$ must be $\in \mathbb{R}^n$. A must have n rows for this to be possible.

2 m: Since $\vec{x} \in \mathbb{R}^m$, A must be $\in \mathbb{R}^{n\times m}$ for the product to be valid.

3 n: $\|\cdot\|$ is a norm on $\mathbb{R}^n$, so b must be $\in \mathbb{R}^n$.

4 m: Simplifying the left side of the equation $(A^*\vec{y})^T\vec{x} = \langle \vec{y}, A\vec{x}\rangle$, we get $\vec{y}^T {A^*}^T \vec{x}$. Since $\vec{x} \in \mathbb{R}^m$ (see solution for 6), ${A^*}^T$ must have m columns for the product ${A^*}^T\vec{x}$ to be valid, which means that A^* must have m rows.

5 n: For the equation $A^*A\vec{x} = A^*\vec{b}$ to be valid, the number of columns in A^* must equal the number of rows in A in order be multiplied.

6 m: Since $A \in \mathbb{R}^{n\times m}$, $\vec{x}$ must be in $\mathbb{R}^m$ for the product $A\vec{x}$ to be valid in the equation $(A^*\vec{y})^T\vec{x} = \langle \vec{y}, A\vec{x}\rangle$.

7 n: Since $A^* \in \mathbb{R}^{m\times n}$, $\vec{y}$ must be in $\mathbb{R}^n$ for the product to be valid in the equation $(A^*\vec{y})^T\vec{x} = \langle \vec{y}, A\vec{x}\rangle$.

8 m: Since $A \in \mathbb{R}^{n\times m}$, all vectors in $N(A)$ or $N(A^*A)$ must be $\in \mathbb{R}^m$.

9 m: For A^*A to be invertible, its nullspace must be trivial, which means that the nullspace of A must also be trivial. This means that $\text{rank}(A)$ must equal the number of columns in A, which is m.

10 m: $(A^*A)^{-1} \in \mathbb{R}^{m\times m}$, $A^* \in \mathbb{R}^{m\times n}$, and $\vec{b} \in \mathbb{R}^n$, so their product must be in $\mathbb{R}^m$

7. Vectors, matrices, and associated operations (12 points)

Consider the following vectors:

$$\vec{a} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 12 \\ 1 \\ 5 \end{bmatrix}, \ \vec{c} = \begin{bmatrix} -8 \\ 0 \\ -4 \end{bmatrix}, \ \vec{d} = \begin{bmatrix} -24 \\ -2 \\ -10 \end{bmatrix}, \ \vec{e} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b}).$$

For each of the following subparts, select all statements that are true. Let $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the usual Euclidean inner product and norm, respectively.

(a) (A) $\left\|\vec{b}\right\| = \sqrt{170}$

(B) $\langle \vec{a}, \vec{b} \rangle < \langle \vec{a}, \vec{c} \rangle < \langle \vec{a}, \vec{d} \rangle$

(C) The sine of the angle between $\vec{a}$ and $\vec{b}$ equals $\frac{59}{\sqrt{170 \times 21}}$

(D) The cosine of the angle between $\vec{a}$ and $\vec{b}$ equals $\frac{59}{\sqrt{170 \times 21}}$

(E) $\vec{a}$ and $\vec{e}$ are orthogonal

(F) $\|\vec{a}\| = 7$

(G) $\vec{a}$ and $\vec{b}$ are orthogonal

Solution: (A), (D), and (E) are true. Let us examine each statement seperately.

(A) $\left\|\vec{b}\right\| = \sqrt{12^2 + 1^2 + 5^2} = \sqrt{170}$. Hence, the statement is True.

(B) $\langle \vec{a}, \vec{b} \rangle = 59$, $\langle \vec{a}, \vec{c} \rangle = -40$, $\langle \vec{a}, \vec{d} \rangle = -118$. False.

(C) False. Wrong application of inner product formula, inner product relates to cosine. See next statement.

(D) $\langle \vec{a}, \vec{b} \rangle = 59$. $\|\vec{a}\| = \sqrt{21}$ and $\|\vec{b}\| = \sqrt{170}$. By applying the formula for an inner product $\langle \vec{a}, \vec{b} \rangle = \|\vec{a}\| \left\|\vec{b}\right\| cos(\theta)$, the statement follows as True.

(E) $\vec{e}$ is obtained by subtracting the projection of $\vec{b}$ onto $\vec{a}$ from $\vec{a}$, and must hence have no component in the direction of $\vec{a}$. The statement is True, and can also be verified computationally.

(F) $\|\vec{a}\| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$. Hence, the statement is False.

(G) Since the inner product of $\vec{a}$ and $\vec{b}$ is non-zero as found in (B), these are not orthogonal. False.

(b) Define the matrix

$$M_1 = \begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{d} \\ | & | & | \end{bmatrix}.$$

Similarly, define

$$M_2 = \begin{bmatrix} | & | \\ \vec{a} & \vec{b} \\ | & | \end{bmatrix} \text{ and } M_3 = \begin{bmatrix} | & | \\ \vec{b} & \vec{d} \\ | & | \end{bmatrix}.$$

(A) M_2 is invertible

(B) The dimensions of M_2 and M_3 are the same

(C) $\text{rank}(M_1) = \text{rank}(M_3)$

(D) M_1 is invertible

(E) $\text{rank}(M_1) = \text{rank}(M_2)$

(F) M_1, M_2, M_3 all have the same number of rows

(G) $\lambda = 0$ is an eigenvalue of M_1

Solution: (B), (E), (F), (G) are True. Consider each statement.

(A) M_2 is not square and hence cannot be invertible. False

(B) The number of rows of each of the matrices are determined by the dimensionality of the columns that comprise them - 3. Hence M_1, M_2, M_3 all have the same number of rows. M_2 and M_3 in addition have the same number of columns, and hence the same dimensions. True

(C) M_1 has two linearly independent columns and has a rank of 2. The second column of M_3 is linearly dependent on the first; hence, M_3's rank is 1. False.

(D) $\vec{d}$ is a multiple of $\vec{b}$; hence, the columns are not linearly independent and the matrix must be non-invertible. False

(E) Both M_1 and M_2 have 2 linearly independent columns; hence, they have the same rank. True

(F) See (B). True

(G) Since M_1 is non-invertible, it has a non-trivial nullspace and hence must have $\lambda = 0$ as an eigenvalue. True

The vectors from before are repeated here for your convenience.

$$\vec{a} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 12 \\ 1 \\ 5 \end{bmatrix}, \ \vec{c} = \begin{bmatrix} -8 \\ 0 \\ -4 \end{bmatrix}, \ \vec{d} = \begin{bmatrix} -24 \\ -2 \\ -10 \end{bmatrix}, \ \vec{e} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b}).$$

(c) Consider the following sets of vectors:

$$S_1 = \{\vec{a}, \vec{b}, \vec{c}\}, \ S_2 = \{\vec{a}, \vec{b}, \vec{d}\}, \ S_3 = \{\vec{a}, \vec{b}\}.$$

(A) $\text{span}(S_1) = \text{span}(S_2)$
(B) $\text{span}(S_1)$ forms a basis for $\mathbb{R}^3$
(C) $\text{span}(S_2) = \text{span}(S_3)$
(D) $\text{span}(S_1) = \mathbb{R}^3$
(E) S_2 forms a basis for some subspace

Solution: (C), (D) are True.

(A) Both S_1 and S_2 contain three vectors. S_1 is a linearly independent set, whereas S_2 is not. The span cannot be the same. False.
(B) A span of a set will include multiples of a vector. Therefore it will not be a linearly independent set. False.
(C) $\text{span}(S_2) = \text{span}(\vec{a}, \vec{b}) = \text{span}(S_3)$ since $\vec{d}$ is a multiple of $\vec{b}$. True.
(D) Any 3 linearly independent vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$. True.
(E) S_2 is not a linearly independent set, so the statement is False.

(d) Let $P, Q \in \mathbb{R}^{m \times m}$ be such that $QP = 0$. Let I denote the $m \times m$ identity matrix.

(A) If $\det(Q) > 0$ then $\det(P) = 0$
(B) $PQ - Q(P - I) = 0$
(C) $\det(P) = 0$
(D) For $\lambda \in \mathbb{R}$, $\det\left(P^T Q^T + \lambda I\right) \neq 0$ if $\lambda \neq 0$

Solution: (A),(D) are True

(A) $\det(QP) = \det(Q)\det(P) = 0$. If $\det(Q) > 0$ then we need $\det(P) = 0$ to satisfy $\det(QP) = 0$. True.
(B) There is no reason for this to be true. False.
(C) All we know is that $\det(QP) = \det(Q)\det(P) = 0$, therefore $\det(P)$ may be non zero. False.
(D) First recall that $\det(A) = \det\left(A^T\right)$ therefore $\det\left(P^T Q^T + \lambda I\right) = \det(QP + \lambda I)$. Now, if $\lambda \neq 0$ then none of the eigenvalues of $QP + \lambda I$ are equal to 0 which implies that $\det\left(P^T Q^T + \lambda I\right) \neq 0$. True.

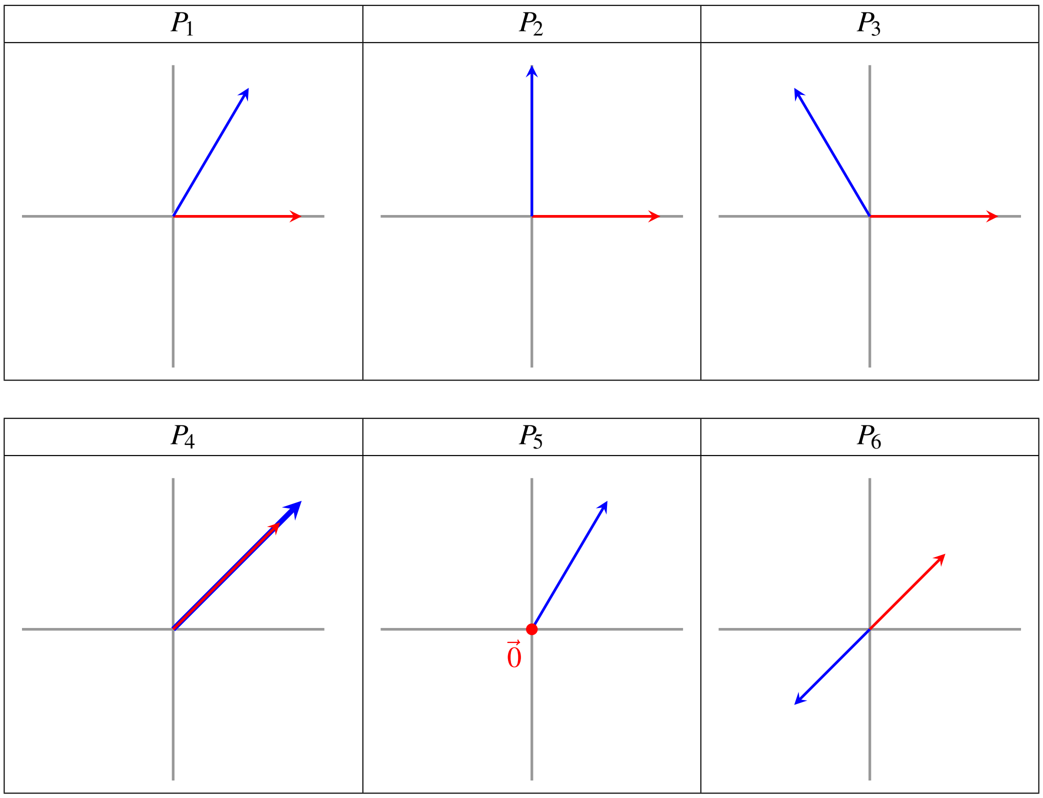
s

8. Visual Vectors (12 points)

(a) Each of the six panels below depicts a pair of two vectors, $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ (one drawn in red, the other in blue). Indicate which of these pairs of vectors, do we have that:

$$|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\|,$$

where $\langle \cdot, \cdot \rangle$ is a given inner product on $\mathbb{R}^2$, and $\|\cdot\|$ is the corresponding norm it induces.



(A) P_5

(B) P_1, P_2, P_3

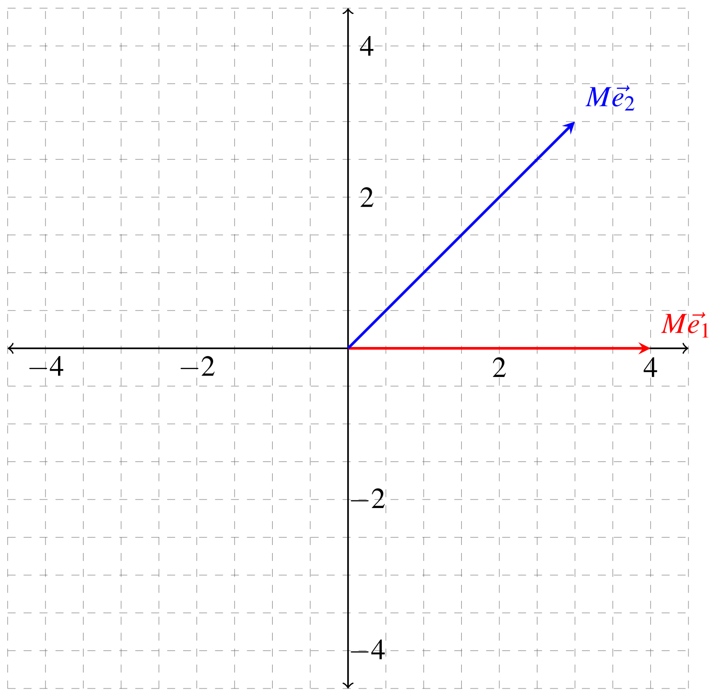
(C) P_2, P_5

(D) P_4, P_5, P_6

(E) P_4, P_6

Solution: (D). Pairs P_4, P_5, and P_6 achieve $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\|\|\vec{y}\|$. This is because these options have $\vec{x}$ and $\vec{y}$ as linearly dependent vectors (same vectors that differ by a scalar multiple). And therefore in the cases of P_4, P_5, and P_6 the equality will be met as (WLOG in these cases taking $\vec{y} = \alpha\vec{x}$): $|\langle \vec{x}, \vec{y} \rangle| = |\langle \vec{x}, \alpha\vec{x} \rangle| = |\alpha||\langle \vec{x}, \vec{x} \rangle| = |\alpha|\|\vec{x}\|^2 = |\alpha|\|\vec{x}\|\|\vec{x}\| = \|\vec{x}\||\alpha|\|\vec{x}\| = \|\vec{x}\|\|\alpha\vec{x}\| = \|\vec{x}\|\|\vec{y}\|$

(b) Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ denote the natural basis vectors in $\mathbb{R}^2$. For a given matrix M, the vectors $M\vec{e}_1$ and $M\vec{e}_2$ are drawn in the plot below. What is the determinant of the matrix M?



(A) 0.0
(B) 6.0
(C) 7.0
(D) -24.0
(E) 24.0
(F) 12.0

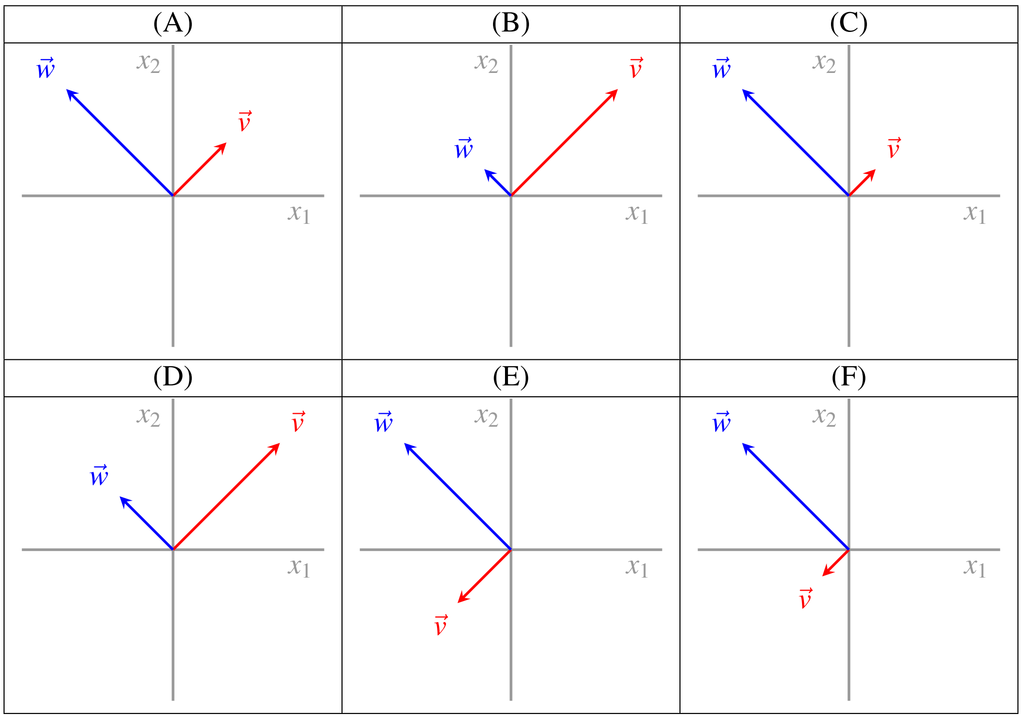
Solution: Answer (F) is correct. Method 1: The area of the resulting parallelogram is 12.
Method 2: Realizing that $M\vec{e}_1$ is the first column of M and $M\vec{e}_2$ is the second column of M. Then M can be written in closed form (aka entries of M can be determined) and the $ad - bc$ expression (where a, b, c, d are the entries of the M matrix) can be used to find the determinant.

(c) You are given the matrices

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Choose the correct illustration of vectors $\vec{v}, \vec{w}$ defined according to

$$\vec{v} = PS^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{w} = PS^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



Solution: (C) is the correct answer: $PS^2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$

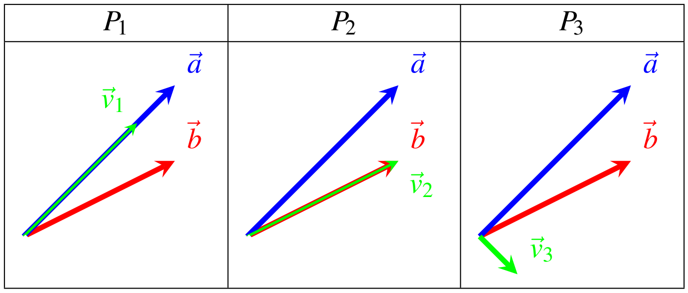
Scaling occurs only in the $\vec{v}$ direction before a rotation.

You can also calculate:

$$\vec{v} = PS^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4\sqrt{2}} \\ \frac{1}{4\sqrt{2}} \end{bmatrix}$$

$$\vec{w} = PS^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

(d) Below you are given three plots P_1, P_2, and P_3 with corresponding vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ drawn on each. For reference, the vectors $\vec{a}, \vec{b}$ are the same in all three plots. Choose the option that correctly expresses the $\vec{v}_i$'s as one of $\vec{b}$, $\text{proj}_{\vec{a}}(\vec{b})$, or $\vec{b} - \text{proj}_{\vec{a}}(\vec{b})$. You should assume that projections are taken with respect to the Euclidean inner product.



(A) $\vec{v}_1 = \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_2 = \vec{b}$, $\vec{v}_3 = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$

(B) $\vec{v}_1 = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_2 = \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_3 = \vec{b}$

(C) $\vec{v}_1 = \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_2 = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_3 = \vec{b}$.

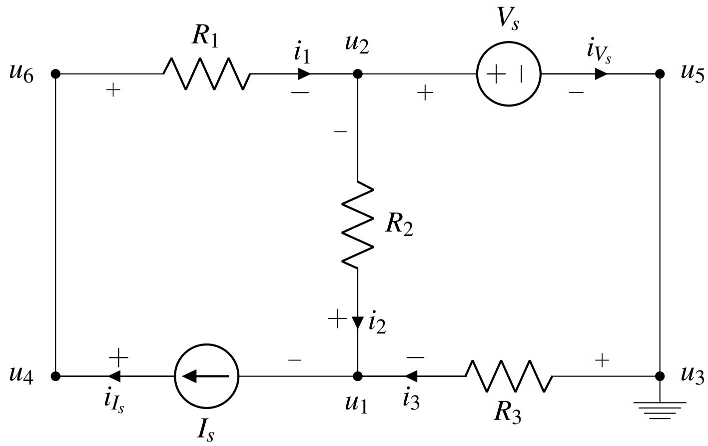
(D) $\vec{v}_1 = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_2 = \vec{b}$, $\vec{v}_3 = \text{proj}_{\vec{a}}(\vec{b})$

(E) $\vec{v}_1 = \vec{b}$, $\vec{v}_2 = \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_3 = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$.

(F) $\vec{v}_1 = \vec{b}$, $\vec{v}_2 = \vec{b} - \text{proj}_{\vec{a}}(\vec{b})$, $\vec{v}_3 = \text{proj}_{\vec{a}}(\vec{b})$

Solution: The correct answer is (A). $\text{proj}_{\vec{a}}\vec{b}$ will be aligned with $\vec{a}$, whereas $\vec{b} - \text{proj}_{\vec{a}}\vec{b}$ will be orthogonal to $\vec{a}$.

9. Nodes and Loops (15 points)



(a) Select all elements of the circuit that have current-voltage labeling that violates passive sign convention:

Solution:

According to the passive sign convention, current flows from the positive to negative polarity across an element. Thus, I_s and R_2 do not follow passive sign convention.

(b) There are more node labelings (u_1, ..., u_6) than necessary. Select all node pairings that describe the same node.

(A) u_6, u_4

(B) u_6, u_2

(C) u_2, u_1

(D) u_5, u_3

(E) u_4, u_1

Solution:

Nodes u_6 and u_4, and u_5 and u_3, are connected by a wire, and are therefore at the same potential. Nodes at the same potential are considered to be the same node.

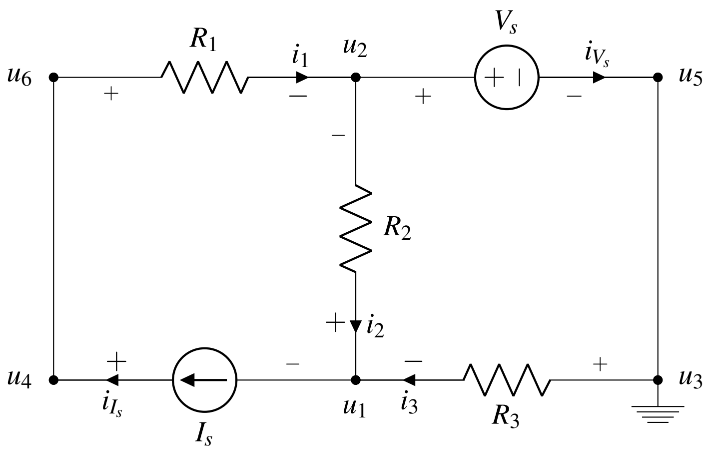
(c) Select the equation for current-voltage relationship of R_1 in terms of resistance, current and node voltages.

Solution:

Using Ohm's law, and noting from the circuit diagram that the voltage drop across $R_1 = u_6 - u_2$, $R_1 = \frac{u_6 - u_2}{i_1}$

Any equation that can be rearranged to become this is fine.

The circuit on the previous page is repeated here for your convenience.



(d) Write the KCL equation for the currents associated with node u_2 in terms of i_1, i_2, i_3, i_4, i_{I_s} and i_{V_s}.

Solution:

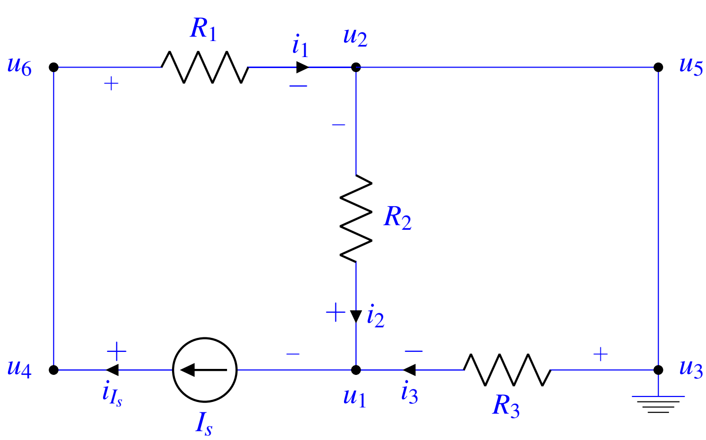
$\sum$ (Currents entering node u_3) = $\sum$ (Currents exiting node u_3)
$i_1 = i_2 + i_{V_s}$
$0 = i_2 + i_{V_s} - i_1$
Any equation that can be rearranged to become this is fine.

(e) Using $V_s = 4\,\text{V}$, $I_s = 5\,\text{A}$, $R_1 = 2\,\Omega$, $R_2 = 3\,\Omega$, and $R_3 = 2\,\Omega$, find the value of i_2.

Solution:

According to the principles of superposition, we set each source to 0 and then sum the resulting currents.

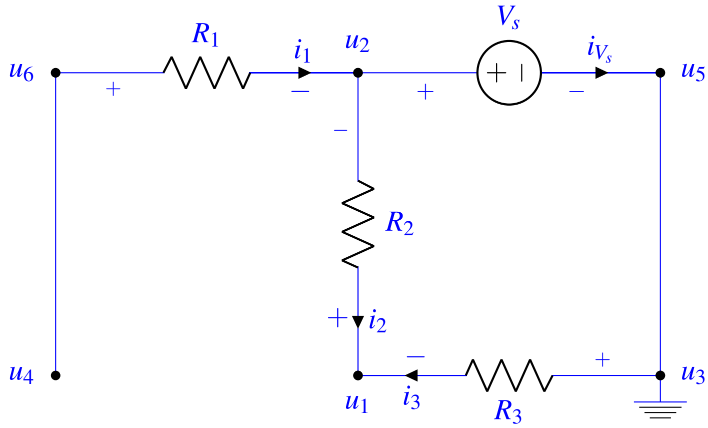
Setting $V_s = 0\,\text{V}$:

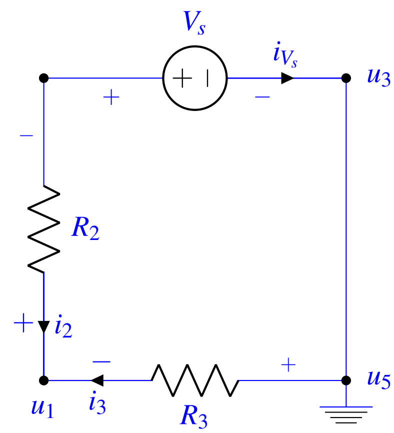


This circuit is a current divider, where I_s reaches u_3 and then splits such that the voltage across each parallel branch remains constant.

$i_a = I_s \frac{R_3}{R_3+R_2} = 5\,\text{A} \frac{2\,\Omega}{2\,\Omega+3\,\Omega} = 2\,\text{A}$

Setting $I_s = 0$:





$i_b = \frac{V_s}{R_3+R_2} = \frac{4\,\mathrm{V}}{2\,\Omega+3\,\Omega} = 0.8\,\mathrm{A}$
$i_2 = i_a + i_b = 2.80\,\mathrm{A}$

10. Scissor, Chisel or Knife: What should I use? (12 points)

Consider the following constrained least squares problem:

$$\min_{\vec{x}} \|A\vec{x} - \vec{b}\| \qquad \text{subject to } \|\vec{x}\|_0 \leq k.$$

Each of the following subparts (a)-(e) specifies A, $\vec{b}$ and k. Your task is to determine which of the three methods you have learned about in EECS16A (Gaussian Elimination, Least Squares and/or OMP), could be used for solving the constrained least squares problem given the problem instance. **You should select all methods that can be reasonably applied to solve the problem.**

For purposes of this problem:

- "Least Squares" is intended to mean evaluating the solution explicitly as:

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}.$$

- You should not rule out using OMP simply because columns of A do not have equal norms. Recall that this was only a simplifying assumption that was made without any loss of generality. Any implementation of OMP would incorporate a preliminary step where the columns of A were rescaled to have norm one.

(a)

$$A = \begin{bmatrix} 0 & 0 & 6 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 7 \\ 9 \\ -1 \end{bmatrix}, \quad k = 4.$$

(A) Gaussian Elimination can be applied.

(B) Least Squares can be applied.

(C) OMP can be applied.

Solution: The A matrix is full column rank, and the system is consistent. All the methods can be applied.
The correct choices are (A),(B),(C)

(b)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 4 \\ 2 & 0 & 2 \\ 9 & 6 & 15 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 9 \\ 12 \\ 6 \\ 45 \end{bmatrix}, \quad k = 3.$$

(A) Gaussian Elimination can be applied.

(B) Least Squares can be applied.

(C) OMP can be applied.

Solution: The system is consistent so Gaussian Elimination can be applied. However, A is column-rank deficient so Least Squares cannot be applied. OMP can be applied as well, but since the column rank is 2, it will terminate after two iterations. Since the cardinality constraint is an inequality and not an equality, the solution that OMP finds with two non-zero entries is still feasible.
The correct choices are hence (A),(C)

(c)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ 2 \\ 0.1 \\ 4 \end{bmatrix}, \quad k = 3.$$

(A) Gaussian Elimination can be applied.

(B) Least Squares can be applied.

(C) OMP can be applied.

Solution: Matrix A is full-column rank. However, as a caveat Least Squares will be computationally faster - this was not a part of the question. Hence, Least Squares and OMP can be applied. However, the system is inconsistent and Gaussian Elimination cannot be applied.
The correct choices are (B),(C).

(d)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0.25 \\ 6 \\ 8 \end{bmatrix}, \quad k = 2.$$

(A) Gaussian Elimination can be applied.

(B) Least Squares can be applied.

(C) OMP can be applied.

Solution: The system is inconsistent, and this rules out Gaussian Elimination. A is column-rank deficient; this rules out Least Squares. However, there are 2 linearly independent columns, so OMP can be applied. We realize that this type of system - consistent, column-rank deficient and sparse - is the canonical example that exhibits OMP's usefulness.
The only correct choice is (C).

11. Best Quadratic Fit (3 points)

You are given vectors $\vec{x}$ and $\vec{y}$ defined as follows

$$\vec{x} = \begin{bmatrix} 1 & 3 & -4 & 3 & -5 \end{bmatrix}^T, \qquad \vec{y} = \begin{bmatrix} 13 & -2 & 7 & -7 & 4 \end{bmatrix}^T.$$

To two decimal places of precision, determine scalars $a, b \in \mathbb{R}$ such that the error in the approximation

$$ax_i^2 + b \approx y_i, \quad i = 1, 2, \ldots, 5$$

is minimized in the sense of least squares (a calculator will be helpful).

(A) $a = 2.62$, $b = -0.95$

(B) $a = -0.95$, $b = 2.62$

(C) $a = 4.33$, $b = -0.11$

(D) $a = -0.95$, $b = 0.00$

(E) $a = -0.11$, $b = 4.33$

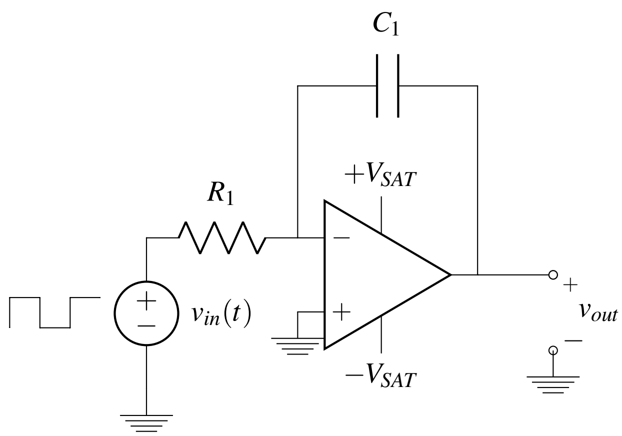
Solution: We apply least squares. The setup is given by:

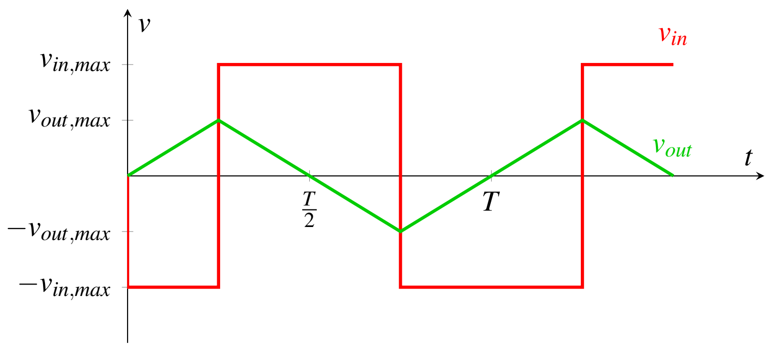
$$\begin{bmatrix} 1 & 1 \\ 9 & 1 \\ 16 & 1 \\ 9 & 1 \\ 25 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \approx \begin{bmatrix} 13 \\ -2 \\ 7 \\ -7 \\ 4 \end{bmatrix}$$

Applying the least squares formula, we get: $\begin{bmatrix} a \\ b \end{bmatrix} = (X^T X)^{-1} X^T \vec{y} = \begin{bmatrix} -0.11 \\ 4.33 \end{bmatrix}$. This corresponds to answer choice **(E)**.

12. Non-Ideal Voltage Source (15 points)

Consider the following circuit and waveforms. Note that the waveforms are not drawn to scale.



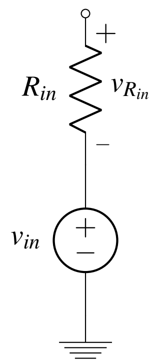


(a) Assume that v_{in} is an ideal voltage source that outputs a square wave with amplitude $v_{in,max} = 5\,\text{V}$ and period $T = 2\,\text{ms}$, and recall that v_{out} will be a triangle wave with peak value $v_{out,max}$. Both waveforms are plotted in the diagram above. If $R_1 = 2\,\text{k}\Omega$, $C_1 = 3\,\mu\text{F}$, and $V_{SAT} = 5\,\text{V}$, what will be the peak value $v_{out,max}$ of our output?

Solution: Define τ as the elapsed time from 0 until the square-wave input first switches to a positive value. Then $\tau = \frac{1}{4}T = 0.25\,\text{ms}$, and

$$v_{out,max} = -\frac{-v_{in,max}}{R_1C_1}\tau = 0.42\,\text{V}$$

(b) Real power supplies cannot act as ideal voltage sources; instead, they have an associated output resistance. Consider the following model of a "real" voltage source. Note: this circuit is separate from part a) for this question.

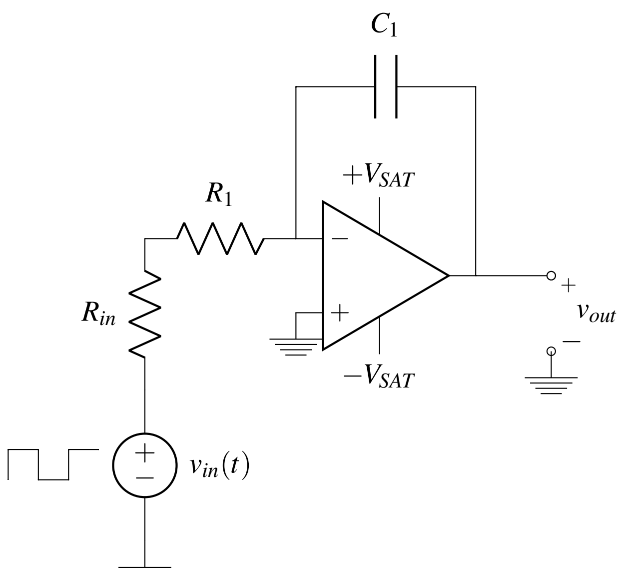


If $R_{in} = 700\,\Omega$, what is $v_{R_{in}}$, the voltage drop across R_{in}?

Solution: This is a "dangling resistor": since it's connected to an open circuit on one side, no current can flow and we have

$$v_{R_{in}} = 0\,\text{V}$$

(c) Now, suppose we modify our earlier circuit model to include this source resistance. Our new circuit diagram is shown below.



Assuming all parameters remain the same as in previous subparts, what is the new peak output voltage $v_{out,max}$?

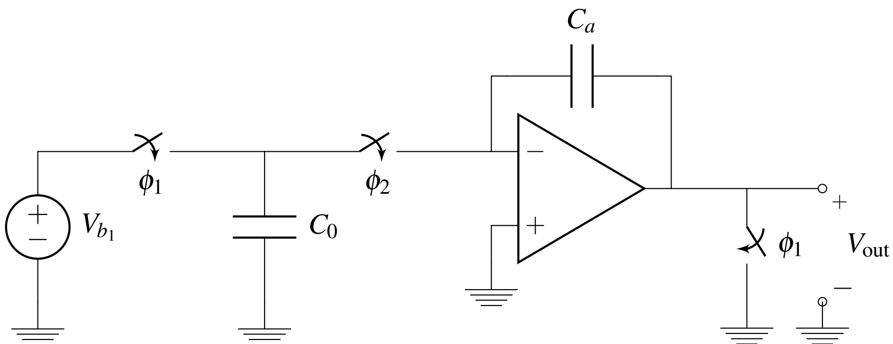
Solution:

$$v_{out,max} = -\frac{v_{in}}{(R_1 + R_{in})C_1}\tau = 0.31\,\text{V}$$

(d) To compensate for the effects of this source resistance, you decide to modify the capacitor value. You can add one capacitor to your circuit, either in series or in parallel with the existing capacitor C_1. Which of the following configurations will result in a triangle-wave output identical to that in part (a)?

Solution: 8.57 µF in series with C_1. This results in an overall capacitance $C_{eq} = 0.74\,C_1$, which compensates for the increased resistance $R_{eq} = 1.35R_1$ and results in the same input-output relationship as in part (a).

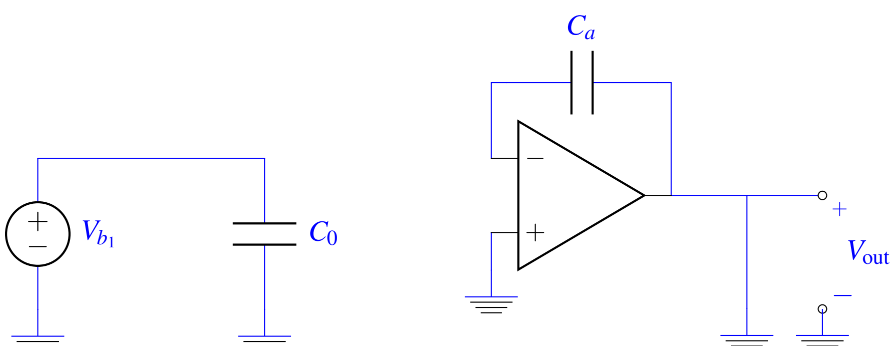
(e) Consider the following circuit:



During phase 1, ϕ_1 is closed and ϕ_2 is open. During phase 2, ϕ_1 is open and ϕ_2 is closed. What is the voltage V_{out} during phase 1 and phase 2? You can assume that a long time has passed in each phase before V_{out} is measured.

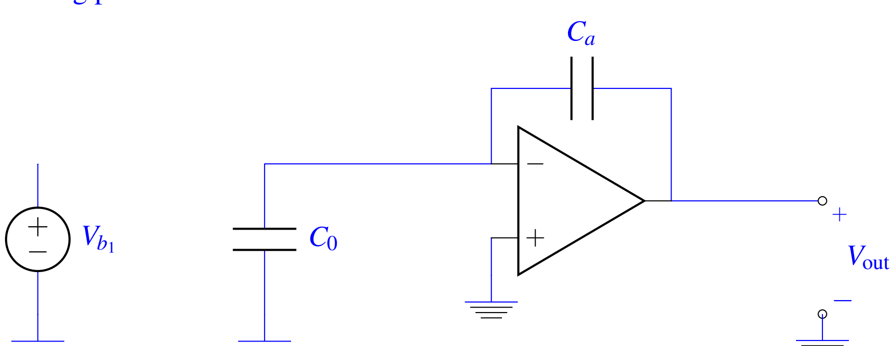
Solution:

During phase 1:



Capacitor C_0 charges from the voltage source: $Q = V_{b_1}C_0$. With ϕ_1 closed, V_{out} is shorted to ground, and therefore $V_{out} = 0\,\text{V}$.

During phase 2:



The charge Q from phase 1 is now shared between C_0 and C_a:

$V_{b_1}C_0 = V_{out}C_a$

$V_{b_1}\frac{C_0}{C_a} = V_{out}$
phase 1: $V_{out} = 0\,\text{V}$. phase 2: $V_{out} = V_{b_1}\frac{C_0}{C_a}$

13. Pagerank with a twist (15 points)

Consider the following pagerank setup that we have encountered before. In this simplified setting, there are only 2 websites - Facebook (F) and Reddit (R). At time $n \geq 0$, denote our state by

$$\vec{x}[n] = \begin{bmatrix} x_F[n] \\ x_R[n] \end{bmatrix}$$

Here, $x_F[n]$ denotes the number of users on Facebook at time n and $x_R[n]$ denotes the number of users on Reddit at time n. The dynamics of the state evolution is modeled as

$$\vec{x}[n+1] = S\vec{x}[n] \quad \text{for } n \geq 0; \qquad S = \begin{bmatrix} w_{FF} & w_{RF} \\ w_{FR} & w_{RR} \end{bmatrix}. \tag{2}$$

However, we do not know the entries of the S matrix, and that is what we are tasked with finding.

(a) Choose the appropriate matrix A from the options for the system of equations below.

$$A \underbrace{\begin{bmatrix} w_{FF} \\ w_{RF} \\ w_{FR} \\ w_{RR} \end{bmatrix}}_{\vec{w}} = \underbrace{\begin{bmatrix} x_F[1] \\ x_R[1] \\ \vdots \\ x_F[T] \\ x_R[T] \end{bmatrix}}_{\vec{b}} \tag{3}$$

(A)

$$\begin{bmatrix} x_F[0] & x_R[0] & x_F[0] & x_R[0] \\ x_F[0] & x_R[0] & x_F[0] & x_R[0] \\ \vdots & \vdots & \vdots & \vdots \\ x_F[T-1] & x_R[T-1] & x_F[T-1] & x_R[T-1] \\ x_F[T-1] & x_R[T-1] & x_F[T-1] & x_R[T-1] \end{bmatrix}$$

(B)

$$\begin{bmatrix} x_F[0] & x_R[0] & 0 & 0 \\ 0 & 0 & x_F[1] & x_R[1] \\ \vdots & \vdots & \vdots & \vdots \\ x_F[T-1] & x_R[T-1] & 0 & 0 \\ 0 & 0 & x_F[T] & x_R[T] \end{bmatrix}$$

(C)

$$\begin{bmatrix} x_F[0] & x_R[0] & 0 & 0 \\ 0 & 0 & x_F[0] & x_R[0] \\ \vdots & \vdots & \vdots & \vdots \\ x_F[T-1] & x_R[T-1] & 0 & 0 \\ 0 & 0 & x_F[T-1] & x_R[T-1] \end{bmatrix}$$

(D)

$$\begin{bmatrix} 0 & 0 & x_F[0] & x_R[0] \\ x_F[0] & x_R[0] & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & x_F[T-1] & x_R[T-1] \\ x_F[T-1] & x_R[T-1] & 0 & 0 \end{bmatrix}$$

(E)

$$\begin{bmatrix} x_F[0] & x_R[0] & -x_F[0] & -x_R[0] \\ -x_F[0] & -x_R[0] & x_F[0] & x_R[0] \\ \vdots & \vdots & \vdots & \vdots \\ x_F[T-1] & x_R[T-1] & -x_F[T-1] & -x_R[T-1] \\ -x_F[T-1] & -x_R[T-1] & x_F[T-1] & x_R[T-1] \end{bmatrix}$$

Solution: For each timestep i, from 2, we have:

$$x_F[i+1] = x_F[i]w_{FF} + x_F[i]w_{RF}$$
$$x_R[i+1] = x_R[i]w_{FF} + x_R[i]w_{RF}$$

Putting this into the desired matrix form, we have that option (C) is correct.

(b) For $T = 1$, is the system (3) consistent? And if yes, does it have a unique solution? Choose the option which best answers the question.

(A) Consistent system, unique solution if and only if $\vec{x}[0]$ is not an eigenvector of S

(B) Consistent system, unique solution if and only if $\vec{x}[0]$ is not a steady state vector of S

(C) Inconsistent system

(D) Consistent system, has a unique solution

(E) Consistent system, has infinite solutions

Solution: Due to the noiseless assumption, the system must be consistent. Since we have 4 unknowns and two equations, the system in part (a) cannot have a unique solution. We need at least 4 equations to obtain a unique solution. The correct answer is thus (E).

(c) For $T = 2$, is the system (3) consistent? And if yes, does it have a unique solution? Choose the option which best answers the question.

(A) Consistent system, has a unique solution

(B) Consistent system, unique solution if and only if $\vec{x}[0]$ is not a steady state vector of S

(C) Consistent system, unique solution if and only if $\vec{x}[0]$ is not an eigenvector of S

(D) Inconsistent system

(E) Consistent system, has infinite solutions

Solution: Extracting the relevant rows from the matrix in part (a) and using the noiseless assumption, our system becomes:

$$\begin{bmatrix} x_F[0] & x_R[0] & 0 & 0 \\ 0 & 0 & x_F[0] & x_R[0] \\ x_F[1] & x_R[1] & 0 & 0 \\ 0 & 0 & x_F[1] & x_R[1] \end{bmatrix} \begin{bmatrix} w_{FF} \\ w_{RF} \\ w_{FR} \\ w_{RR} \end{bmatrix} = \begin{bmatrix} x_F[1] \\ x_R[1] \\ x_F[2] \\ x_R[2] \end{bmatrix}$$

For the system to have a unique solution, it must be full column (and hence also row) rank. This will only happen if $\vec{x}[1]$ is not a multiple of $\vec{x}[0]$ i.e if $\vec{x}[0]$ is not an eigenvector of S. Option (C) is correct.

(d) Now, suppose we do not observe the states $\vec{x}[n]$ directly, but instead we are provided with imperfect estimates of the system state up to timestep T. That is, we are given the collection of vectors $\{\vec{y}[0]\ldots\vec{y}[T]\}$, where $\vec{y}[i]$ is a noisy observation of the state $\vec{x}[i]$ at time i. Hence, we replace all values of $x_F[i]$ and $x_R[i]$ with $y_F[i]$ and $y_R[i]$, respectively, in the definitions of $A, \vec{b}$ in (3). You discover that the resulting system of equations is inconsistent and that matrix A has linearly independent columns.

What is the best procedure to find $\vec{w}^*$ that minimizes the error $\left\|A\vec{w} - \vec{b}\right\|$?

(A) Use Least Squares, $\vec{w}^* = (A^T A)^{-1} A^T b$

(B) Use Orthogonal Matching Pursuit, $\vec{w}^* = (S^T S)^{-1} S^T b$

(C) Use Least Squares, $\vec{w}^* = (S^T S)^{-1} S^T b$

(D) Use Orthogonal Matching Pursuit, $\vec{w}^* = (A^T A)^{-1} A^T b$

(E) Use Gaussian Elimination

Solution: Gaussian Elimination cannot give approximate solutions of inconsistent systems. Since we are not given the sparsity level of our w, OMP would not be the best approach here. We could assume that w has no zero elements and run OMP. But that would give the same solution as least squares and be computationally wasteful. Least Squares is the best option here. Option (A) is the choice; S contains the entries of w and (C) can thus be eliminated.

(e) You apply the procedure you selected in the previous part and obtain

$$\vec{w}^* = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.7 \\ 0.5 \end{bmatrix}.$$

Now that you have an approximate solution for the unknown state transition matrix S, you can attempt to answer questions about the pagerank evolution in System (2). Does System (2) have a steady state and does $\lim_{n\to\infty} S^n \vec{x}[n]$ converge for all choices of $\vec{x}[0]$? Choose the option which best answers the question.

(A) Does not have a steady state, may not converge

(B) Has a steady state, may not converge

(C) Has a steady state, always converges

(D) Does not have a steady state, always converges

Solution: From $\hat{w}$ provided, we find that the state transition matrix, S in 2 is

$$S = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$$

We find that this S has eigenvalues -0.2 and 1. Hence, the system both has a steady state and converges. The correct answer is (C).

14. Matching Pursuit (9 points)

Consider the constrained least squares problem

$$\min_{\vec{x}\in\mathbb{R}^5} ||M\vec{x}-\vec{b}|| \quad \text{subject to: } \|\vec{x}\|_0 \le k.$$

In the above, the matrix M and the vector $\vec{b}$ are given by:

$$M = \begin{bmatrix} \sqrt{1/5} & 0 & \sqrt{1/2} & \sqrt{1/4} & \sqrt{1/3} \\ \sqrt{4/5} & 1 & \sqrt{1/2} & \sqrt{3/4} & \sqrt{2/3} \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}.$$

(a) If we ran Matching Pursuit for one iteration, which coordinates of the resulting solution $\vec{x} = [x_1, x_2, \dots, x_5]^T$ would be nonzero? Select all that apply.

(A) x_1
(B) x_2
(C) x_3
(D) x_4
(E) x_5

Solution: (E).
Notice that if we compute all the inner products for the columns of the matrix M we have that $\vec{m}_5 = [\sqrt{1/3}, \sqrt{2/3}]^T$ is the vector that maximizes $|\langle \vec{m}_i, \vec{e}\rangle|$. Therefore, after one iteration of the algorithm we will have identified the component of $\vec{e}$ along $\vec{m}_5$ which means that $x_5 \neq 0$.

(b) If we ran Matching Pursuit for two iterations, which coordinates of the resulting solution $\vec{x} = [x_1, x_2, \dots, x_5]^T$ would be nonzero? Select all that apply.

(A) x_1
(B) x_2
(C) x_3
(D) x_4
(E) x_5

Solution: (B) and (E).
We must have x_5 nonzero since that is set in the first iteration as we saw above. After this, the next vector that maximizes $|\langle \vec{m}_i, \vec{e}\rangle|$ (excluding $\vec{m}_5$) is $\vec{m}_2$. Therefore, in the second iteration we identify the component of $\vec{e}$ along $\vec{m}_2$ which means that x_2 is nonzero.

(c) If we ran Orthogonal Matching Pursuit for two iterations, what would be the norm of the resulting residual $\vec{e} = \vec{b} - M\vec{x}$? Choose the option which best answers the question.

(A) $||\vec{e}||^2 = 3$
(B) $||\vec{e}||^2 = 1$
(C) $||\vec{e}||^2 = 4$
(D) $||\vec{e}||^2 = 0$
(E) $||\vec{e}||^2 = 2$

Solution: (D).
Notice that any subset of two of M's columns are linearly independent: they are not multiples of each other. On the first iteration of OMP, one column will be selected, then on the second iteration, another will be selected. Since these columns are linearly independent and $\vec{b}$ is in the span of the two columns selected ($\vec{b} \in \mathbb{R}^2$), the least squares solution will yield an error of zero.

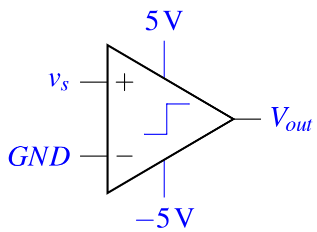
15. Building a Noise-Resistant Comparator (18 points)

In many sensing applications in which are you trying to distinguish between two states (for example determining if a light is on or off, or if it's hot or cold) the output of a comparator is often used as a function of time. However, in many situations there is significant noise in the circuit which can lead to false transitions between the states. In this problem, we will explore the pitfalls of simply using a comparator and see how to remedy the situation.

We will be analyzing a hypothetical sensing application where the on-state occurs when the input voltage is positive and the off-state occurs when the input voltage is negative.

(a) Select the comparator circuit that would output V_{DD} when the system is in the on-state and V_{SS} when it is in the off state. Let v_s be the input signal.

Solution:



From the problem description, we would like the output to be V_{DD} when the input voltage is positive (on-state) and V_{SS} when the input voltage is negative (off-state). The right most node on the comparator is the ouput, which is V_{out}. The inputs of a comparator are the terminals marked (+) and (-). v_s is one input, and by setting the second input to GND, we are effectively comparing whether v_s is greater or less than 0, which compares whether the input voltage is positive or negative. By setting V_{DD} to 5 V and V_{SS} to -5 V, $V_{out} = V_{DD}$ in the on-state and $V_{out} = V_{SS}$ in the off-state.

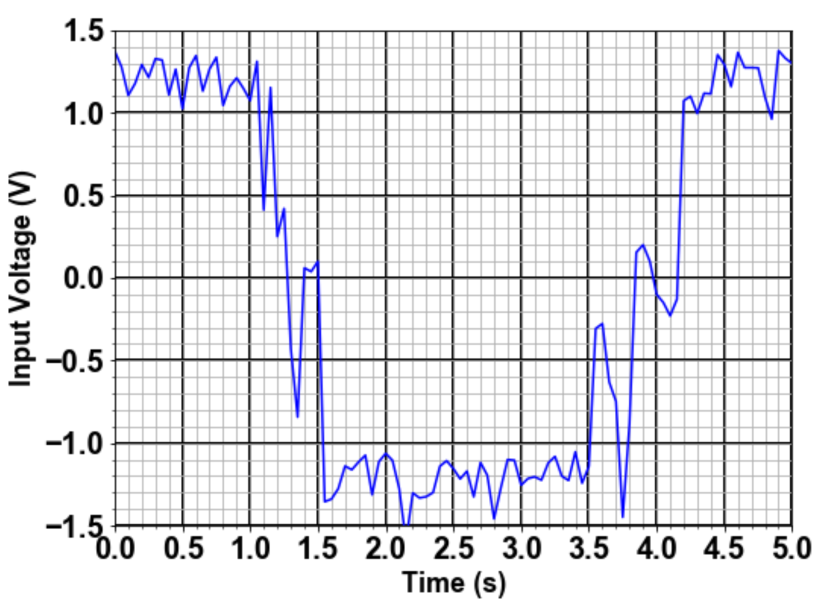


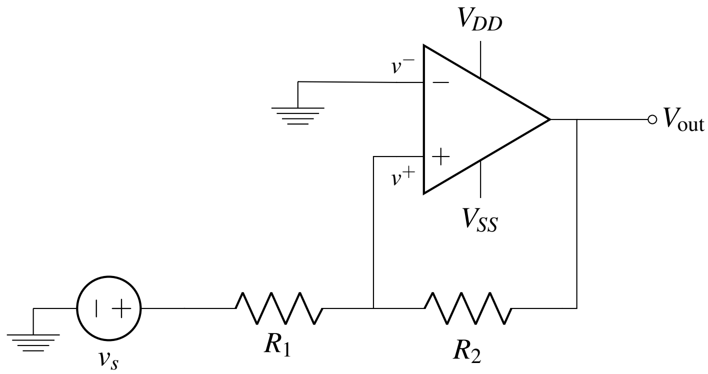
Figure 1: Noisy input signal

(b) Now consider the above noisy input signal. Determine the total number of times the output of the comparator switches between V_{DD} and V_{SS}.

Solution:

Each time the input voltage (y-axis of Fig. 1) changes from negative to positive or positive to negative, the output of the comparator will switch (as described in part a). In Fig. 1, we can count 6 transitions- at t = 1.3, 1.4, 1.5, 3.8, 4.0, 4.2s.

(c) Seeing that a simple comparator circuit will not do the job, you ask a Berkeley EECS student to help. They provide you with the following circuit and says it should make your comparator circuit more resistant to noise.



Determine v^+ as a function of v_s and V_{out} and other passive components.

Solution:

Applying KCL at the v^+ node, we get:

$$0 = \frac{v^+ - v_s}{R_1} + \frac{v^+ - V_{out}}{R_2}$$
$$v^+ = \frac{v_s R_2 + V_{out} R_1}{R_1 + R_2}$$

(d) Assume that V_{out} was at V_{SS}. What voltage would v_s have to be in order to for V_{out} to change to V_{DD}?

Solution:

If $V_{out} = V_{SS}$, that implies that $v^+ < 0\,\text{V}$. In order to switch to V_{DD}, we need $v^+ > 0\,\text{V}$. Thus, we take the result from the previous part and set $v^+ = 0$ (as that is the transition point) and solve for v_s.

$$0 = \frac{V_s R_2 + V_{SS} R_1}{R_1 + R_2}$$
$$V_s = -\frac{V_{SS} R_1}{R_2}$$

(e) Now assume that V_{out} was V_{DD}. What voltage would v_s have to be in order for V_{out} to change to V_{SS}?

Solution:

Similarly, if $V_{out} = V_{DD}$, that implies that $v^+ > 0\,\text{V}$. In order to switch to V_{SS}, we need $v^+ < 0\,\text{V}$. Thus we simply repeat the analysis from the previous case, except now $V_{out} = V_{DD}$:

$$0 = \frac{v_s R_2 + V_{DD} R_1}{R_1 + R_2}$$
$$V_s = -\frac{V_{DD} R_1}{R_2}$$

What we have shown here is that the condition on v_s for switching the output from V_{DD} to V_{SS} or vice versa is no longer simply changing signs of v_s as it was for the comparator circuit. This circuit introduces thresholds which if set correctly can remove substantial noise from the output.

(f) From the previous parts, we see that we have different input threshold voltages depending on whether the comparator is outputting $V_{DD} = 5\,\text{V}$ or $V_{SS} = -5\,\text{V}$. Now pick values for R_1 and R_2, which will remove the noise from the circuit. Choose values for R_1 and R_2 that set input threshold voltages of exactly ± 1V.

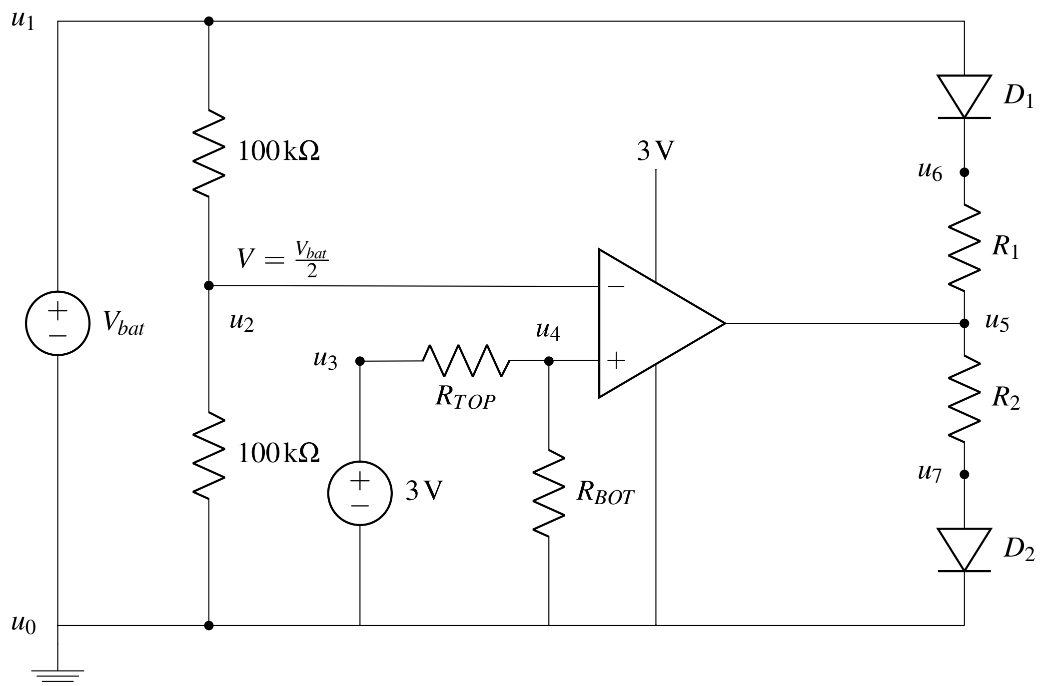
Solution:

From part (b), there are a lot of oscillations as the system is changing states. From the waveform, we see that if we pick thresholds of ± 1V, we can avoid the oscillation problem. For thresholds of ± 1V, we see that we require $\frac{R_1}{R_2} = \frac{1}{5}$. Thus, we can simply pick $R_1 = 1\,\text{k}\Omega$ and $R_2 = 5\,\text{k}\Omega$.

16. Circuit Design for COVID-19 (9 points)

After learning that the novel coronavirus pandemic has resulted in a global shortage of personal protective equipment, you decide to team up with other CS, EE, and ME Berkeley engineering students to build a special kind of respirator known as a PAPR, which is short for powered air purifying respirator. Your job is to quickly prototype the electrical parts of the design to demonstrate basic viability.

(a) (3 points) Battery Status Indicator: You start by designing a circuit that will indicate whether the battery is sufficiently charged, or if the battery is low and needs to be recharged. To do this, you sketch out the following circuit that includes a comparator with 10 mV of built-in hysteresis, a couple of LEDs, some resistors, and a voltage source:



The green LED operates with a forward voltage, $V_{f,grn} = 2.5\,\mathrm{V}$, and current $I_{f,grn} = 100\,\mathrm{mA}$, so we need to determine the value of its current limiting resistor R_{GRN}. Similarly, the red LED operates with a forward voltage, $V_{f,red} = 2.5\,\mathrm{V}$, and current $I_{f,red} = 50\,\mathrm{mA}$, so we need to determine the value of its current limiting resistor R_{RED}.

Your team is using a Lithium-ion battery whose output, V_{bat}, ranges from 3.2 V when discharged to 4.2 V when charged. You have decided that 3.45 V is a reasonable threshold to switch between the green and red LEDs (representing 25% battery charge). You need to ensure that LED never draws more than I_f.

i. (1 point) What is the correct mapping:

Solution:

The comparator in this circuit has two inputs, $\frac{V_{bat}}{2}$ and the threshold voltage for the battery. When $\frac{V_{bat}}{2}$ < threshold voltage, we want the LED to be red- the conventional color for low battery. When $v^- < v^+$, V_{DD} is output (3 V). This will turn on D_2, so we set D_2 to red. Therefore, D_1 is green.

ii. (1 point) What should be the value of R_{GRN}?

Solution: $R_{GRN} = \frac{V_{batmax} - V_{f,grn}}{I_{f,grn}} = \frac{4.2\,\mathrm{V} - 2.5\,\mathrm{V}}{100\,\mathrm{mA}} = 17\,\Omega$

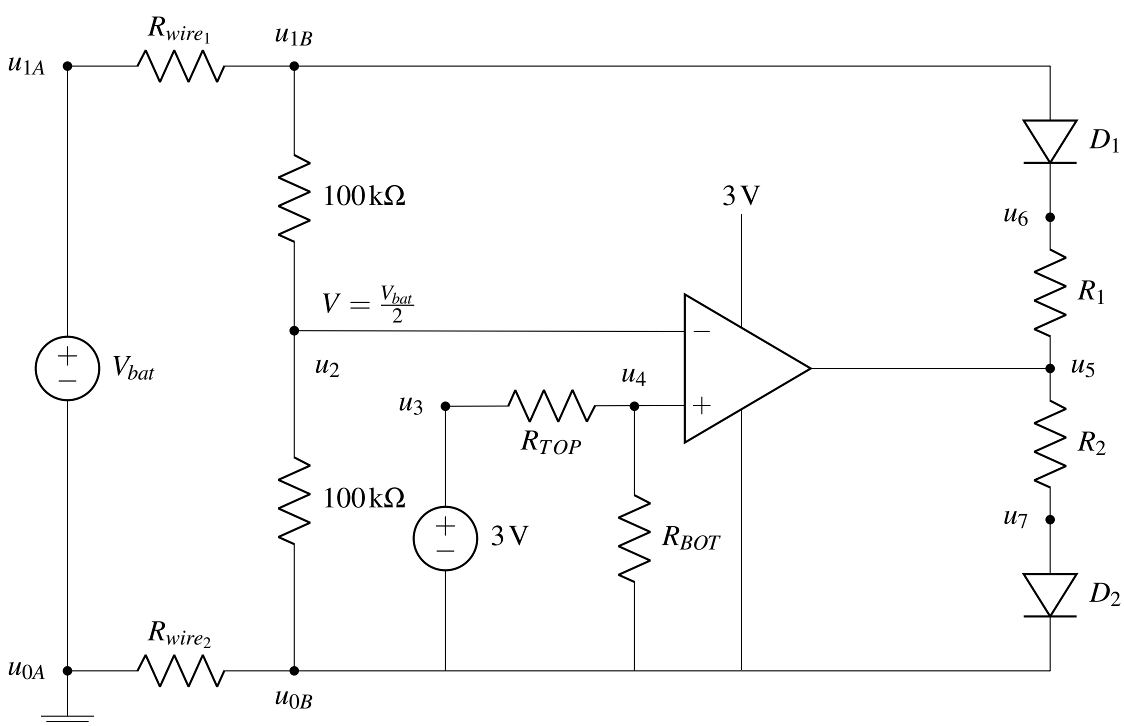
iii. (1 point) What should be the value of R_{TOP}? (Assuming $R_{BOT} = 100\,\mathrm{k}\Omega$)

Solution:
Node u_4 should equal the threshold voltage we want to compare V_{bat} to, but because the 100 kΩ-100 kΩ voltage divider leading into the negative input of the op-amp reduced the comparator negative input terminal to $\frac{V_{bat}}{2}$, u_4 needs to be scaled accordingly by a factor of $\frac{1}{2}$.
$u_4 = \frac{3.45\,\text{V}}{2} = 3\,\text{V} \cdot \frac{R_{BOT}}{R_{BOT}+R_{TOP}}$
Plugging in 100 kΩ for R_{BOT}, and solving for R_{TOP}:
$R_{TOP} = \frac{2.55}{3.45} R_{BOT}$

$R_{TOP} = 73.9\,\text{k}\Omega$

(b) (3 points) Debugging Indicator Instability: You breadboard your design in the lab and get it working! Upon further testing, your team members note that the red and green LEDs seem to rapidly flash back and forth when the battery level is close to 25% charge. You never noticed this on the bench when you simulated the battery draining by quickly sweeping the voltage from 4.2 V to 3.2 V, so you walk over to check out what is going (while maintaining a safe social distance).

You notice that your mechanical engineering friends have connected your circuit to the battery using a 10 ft length of 30 gauge wire (ρ = 103 $m\Omega$/ft, R = R_{wire_1}) between the positive terminal of V_{bat} (node u_{1A}) and the rest of the circuit (node u_{1B}) but they are using a 6 inch jumper cable made of 22 gauge wire (ρ = 16 $m\Omega$/ft, R=R_{wire_2}) to connect the negative terminal of the battery (node u_{0A}) and the rest of the circuit is ground (node u_{0B}).



You suspect that there might be a sudden change in the voltage drop across these wires that reduces the voltage (nominally $\frac{V_{bat}}{2}$) at the V^- input of the comparator when toggling between the two LEDs. Ignoring any hysteresis provided by the comparator, you decide to add your own hysteresis to try to solve the flashing problem.

i. (1 point) Between which two nodes in the circuit would you add a resistor, R_{HYST}, to add hysteresis to the comparator:

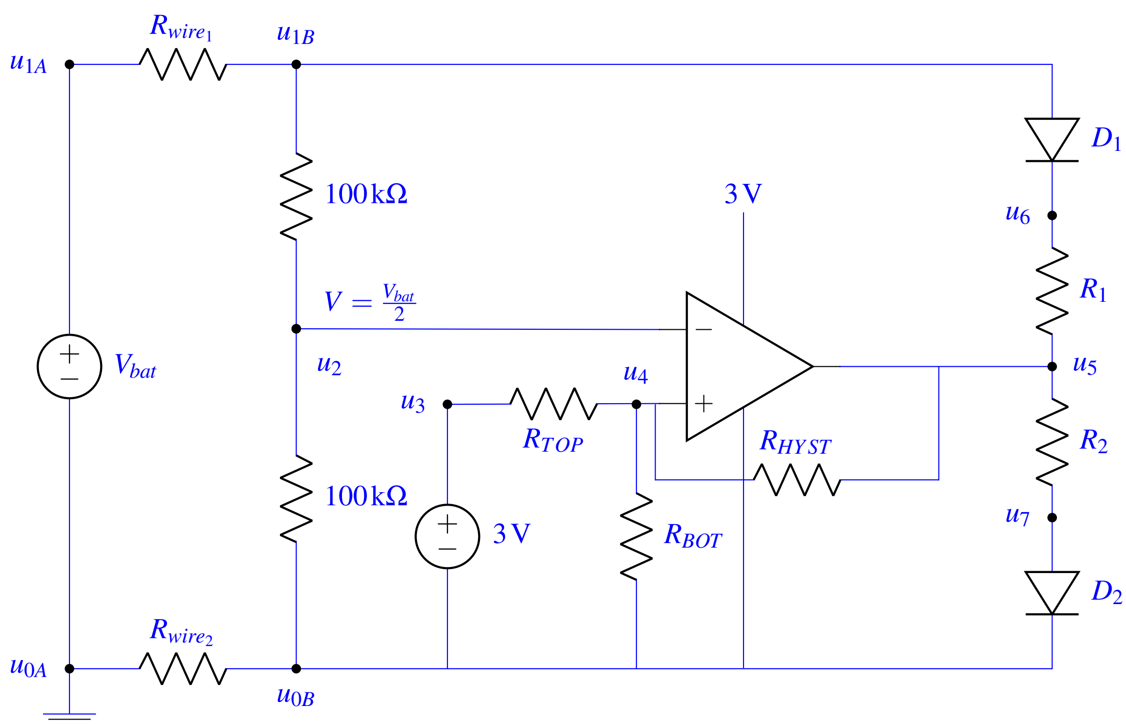
Solution:
The use of a 10-ft long cable (R_{wire_1}) is causing a voltage drop to the negative input of the comparator. Adding a resistor between $V_{OUT}(u_5)$ and $v^+(u_4)$ will add hysteresis to the positive input.

ii. (1 point) How much hysteresis (in volts) do you need to add to stop the rapid flashing between red and green LEDs around the 25% threshold?

Solution:
V_{HYST} = (10 ft) * (0.103 ohm/ft) * (I_{GRN} - I_{RED}) = 1.03 Ω * 50 mA = 52 mV

iii. (1 point) Select the maximum value of R_{HYST} from this list, that will provide the required margin (ignoring any hysteresis provided by the comparator)?

Solution:



We solve this circuit with superposition, first setting the 3 V source to 0 V, and then setting the output of the comparator to 0 V:

$$\frac{R_{BOT}//R_{TOP}}{(R_{BOT}//R_{TOP})+R_{HYST}}V_{bat} + \frac{R_{HYST}//R_{BOT}}{(R_{HYST}//R_{BOT})+R_{TOP}}$$

Hysteresis is related to the change of the comparator output, so we ignore the superposition term related to the highly stable 3 V source and focus on the V_{bat} term:

$$V_{HYST} = \frac{R_{BOT}//R_{TOP}}{(R_{BOT}//R_{TOP})+R_{HYST}}V_{bat}$$

Trying $R_{HYST} = 1\,\mathrm{M}\Omega$, we solve to find the resulting hysteresis:

$$V_{HYST} = \frac{100\,\mathrm{k}\Omega//73.9\,\mathrm{k}\Omega}{(100\,\mathrm{k}\Omega//73.9\,\mathrm{k}\Omega)+1\,\mathrm{M}\Omega}3.45\,\mathrm{V} = 140\,\mathrm{mV}$$

1 MΩ is the maximum resistor available that will provide the necessary hysteresis.

(c) Brainstorming Other Ideas: You describe the problem to some of your EECS 16A friends and they all offer their own ideas. You need to eliminate the bad ideas to preserve your time to test the good ones. Select ALL of the following ideas that WILL NOT solve the flashing problem:

(A) Increase the value of R_{RED}

(B) Increase the value of R_{BOT} or decrease the value of R_{TOP}

(C) Swap the inputs to v^+ and v^- and the locations of D1 and D2

(D) Use green and red LEDs that draw the same current

(E) Use a green LED that draws much more current than the red LED

(F) Swap the inputs to v^+ and v^- and the polarities of D1 and D2

(G) Use a larger diameter wire for the 10 ft run

(H) Add a resistance in series with the jumper wire that is about the same as the 10 ft wire

Solution:

Will not solve the flashing problem:

Increase the value of R_{BOT} or decrease the value of R_{TOP}

- This will change the threshold voltage by adjusting the voltage divider at u_4, but not affect the hysteresis.

Swap the inputs to v^+ and v^- and the polarities of D_1 and D_2

- With v^+ and v^- reversed, the comparator would output 3 V when $\frac{V_{bat}}{2} > 3.45\,\text{V}$. By reversing the polarities of the LEDs, they will not turn on.

Use a green LED that draws much more current than the red LED

- This difference in the currents between LEDs is what causes hysteresis, due to a sudden change in the current being drawn, and therefore a drop in voltage across R_{wire_1}. Having LEDs with greater current difference will increase hysteresis.

Increase the value of R_{RED}

- Limiting the red LED current will increase (I_{GRN} - I_{RED}), and increase the hysteresis.

Will solve the flashing problem:

Use a larger diameter wire for the 10 ft run

- Increasing the wire diameter will reduce the resistance of the 10ft wire, and reduce the voltage drop that occurs because of the wire's resistance. This will reduce hysteresis.

Add a resistance in series with the jumper wire that is about the same as the 10 ft wire

- By equalizing the resistances leading into each input, an equal amount of voltage will be dropped across both comparator inputs. Because the comparator measure one input relative to another, dropping equal voltage across both inputs will not affect the comparator output.

Use green and red LEDs that draw the same current

- As shown in (ii) of this section, the hysteresis is proportional to the difference in currents from the red and green LEDs. If both LEDs drew the same current, there would be no difference between them and therefore no hysteresis.

Swap the inputs to v^+ and v^- and the locations of D_1 and D_2
- When the red LED switches on, it will draw less current from V_{bat} than the green LED did, and therefore cause a smaller voltage drop across R_{wire_1} and lower hysteresis.