Midterm Solutions

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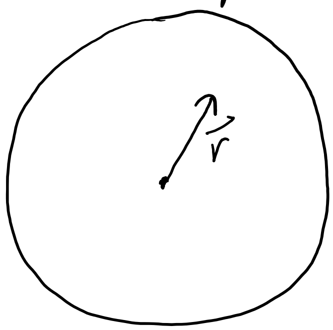
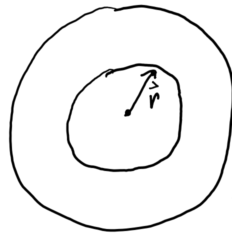
Problem 1:

a) Let's first see what the field looks like at an arbitr. point $\vec{r}$ within a sphere of charge density ρ:

we want $\vec{E}(\vec{r})$



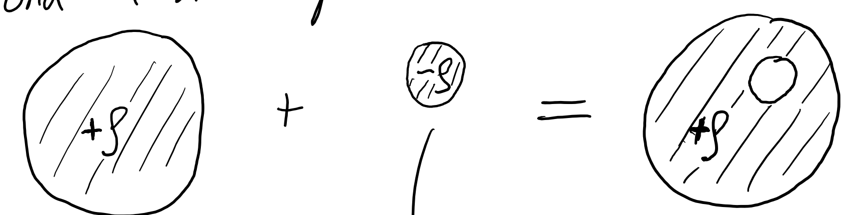
$\Rightarrow$



$$\overset{\text{Gauss}}{\Rightarrow} |E(\vec{r})| 4\pi |\vec{r}|^2 = \frac{4}{3\varepsilon_0}\pi |\vec{r}|^3 \rho$$

$$\Rightarrow \vec{E}(\vec{r}) = \frac{\vec{r}\rho}{3\varepsilon_0}$$

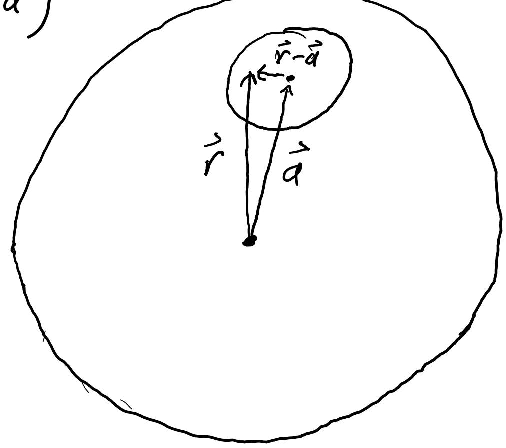
The setup in the problem can be recreated using a full sphere of radius R_1 with ρ and a small sphere of radius R_2 with $-\rho$



$\leftarrow$ intuition

located at 0

located at $\vec{a}$ (direction doesn't actually matter as long as $|\vec{a}| = a$)



$$\Rightarrow E_{total}(\vec{r}) = \underbrace{\frac{\vec{r}\rho}{3\varepsilon_0}}_{\text{big full ball}} + \frac{(\vec{r}-\vec{a})(-\rho)}{3\varepsilon_0}$$

use formula from above, just with coordinate shift for the small ball

$$= \boxed{\frac{\vec{a}\rho}{3\varepsilon_0} = \vec{E}(\underset{\text{within cavity}}{\vec{r}})}$$

E-field within the cavity is constant

i.e. the E-field within the cavity is constant and pointing in the direction connecting the big ball's and the cavity's centers.

b) potentials can also be added

set $\phi=0$ at ∞.

Let's first ignore the specific setup and get a general formula for the potential at an arbitrary point r inside a charged spere:

$$\phi = \int_\infty^r -E\,dr = \int_\infty^R -E_{out}\,dr + \int_R^r -E_{in}\,dr = -\frac{\rho}{3\varepsilon_0}\left[\int_\infty^R \frac{R^3}{r^2}dr + \int_R^r r\,dr\right]$$

$$E_{out} = \frac{1}{4\pi\varepsilon_0}\frac{\rho\frac{4}{3}\pi R^3}{r^2} = \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2} \qquad E_{in} = \frac{\rho r}{3\varepsilon_0}$$

$$= -\frac{\rho}{3\varepsilon_0}\left[-R^2 + \frac{1}{2}(r^2 - R^2)\right]$$

$$= \frac{\rho}{6\varepsilon_0}\left(3R^2 - r^2\right) =: V(r)$$

now we can add the two potentials from the big and the small sphere:

$$V_{\substack{\text{center}\\ \text{of cavity}}} = \underbrace{V_{\rho R_1}(a)}_{\text{big ball}} + \underbrace{V_{-\rho R_2}(0)}_{\text{small ball}} = \frac{\rho}{6\varepsilon_0}\left(3R_1^2 - a^2\right) + \frac{(-\rho)}{6\varepsilon_0}\left(3R_2^2 - 0\right)$$

$$= \frac{\rho}{6\varepsilon_0}\left(3R_1^2 - a^2 - 3R_2^2\right)$$

$$= \boxed{\frac{\rho}{\varepsilon_0}\left(\frac{1}{2}(R_1^2 - R_2^2) - \frac{a^2}{6}\right) = V_{\substack{\text{center}\\ \text{of cavity}}}}$$

$$= \boxed{\frac{J}{\varepsilon_0}\left(\frac{1}{2}(R_1^2 - R_2^2) - \frac{a^2}{6}\right) = V_{\substack{\text{center}\\ \text{of cavity}}}}$$

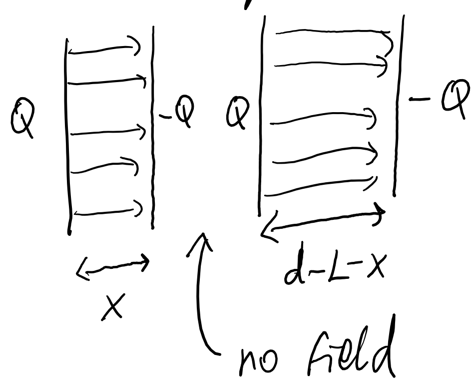
Problem 2:

a) The metal slab is a conductor

$\Rightarrow$ no field in II

the surface of the metal will arrange such that we have $-Q$ on the left and $+Q$ on the right, effectively giving us 2 capacitors:

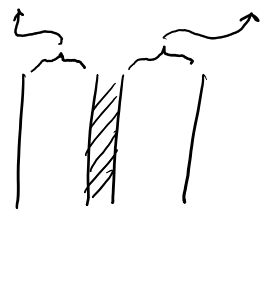
$$E = \frac{\Delta\phi}{\ell} = \frac{Q}{C\ell} = \frac{Q\,\ell}{\ell A \varepsilon_0} = \frac{Q}{A\varepsilon_0}$$



$$\Rightarrow E_I = \frac{Q}{A\varepsilon_0} = E_{III}$$

$$\Rightarrow \boxed{\begin{aligned} E_I &= \frac{Q}{A\varepsilon_0} \\ E_{II} &= 0 \\ E_{III} &= \frac{Q}{A\varepsilon_0} \end{aligned}}$$

now get the potential difference: $\Delta\phi = Ex + E(d-L-x)$



$$\Rightarrow \boxed{\Delta\phi = \frac{Q}{A\varepsilon_0}(d-L)}$$

b) Work = difference in energy between the two setups

$$\ldots U = \frac{1}{2}\frac{Q^2}{\ldots} = \frac{1}{2}\frac{Q^2 d}{\ldots}$$

b) Work = difference in energy ...

without metal: $U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q^2 d}{A\varepsilon_0}$

energy within field of cap.

with metal: $U = \frac{1}{2} Q^2 \left(\frac{1}{C_I} + \frac{1}{C_{III}} \right) = \frac{1}{2A\varepsilon_0} Q^2 (x + d - L - x)$

$$= \frac{1}{2}\frac{Q^2}{A\varepsilon_0}(d-L)$$

$$\Rightarrow \Delta U = \boxed{\frac{1}{2}\frac{Q^2}{A\varepsilon_0} L = \text{Work}}$$

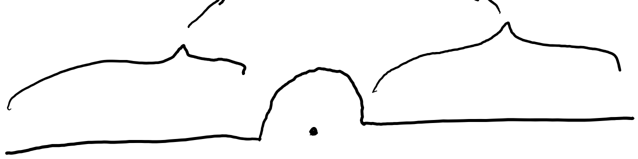
Problem 3:

Use Biot-Savart: $\vec{B}(\vec{r}) = \int_{\text{wire}} \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}'}{r'^2}$

in our case $\vec{r} = 0$ and $\vec{r}'$ is the vector connecting O and the point of the wire over which we are integrating. $d\vec{\ell}$ points in the direction of I.

here we see that $d\vec{\ell} \parallel \hat{r}$

$\Rightarrow$



$\Downarrow$ value of integrand in these parts of the wire is 0

$$\Rightarrow B = \frac{\mu_0 I}{4\pi} \int_{\frown} \frac{d\vec{\ell} \times \hat{r}'}{r'^2} = \frac{\mu_0 I}{4\pi} \int_{\frown} \frac{d\ell}{r'^2} = \frac{\mu_0 I}{4\pi r'^2} \underbrace{\int_{\frown} d\ell}_{= \frac{1}{2}(2\pi r')}$$

$d\vec{\ell} \perp \hat{r}'$ on the half circle

r' is constant $= 1\,\text{cm}$ on the circle

= half a circumference

$$= \frac{\mu_0 I}{4 r'} = \boxed{\frac{\mu_0 \, 1\text{A}}{4\,\text{cm}} = B}$$ =3,14x10^-5 Tesla

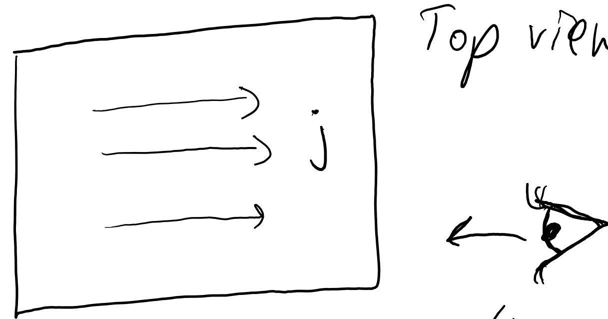
The field points into the page.

Also, if you don't wanna use biot savart you could basically just use the formula of a loop of charge, take half of that and be done. In other words you could have done this in 2 lines.

Problem 4.

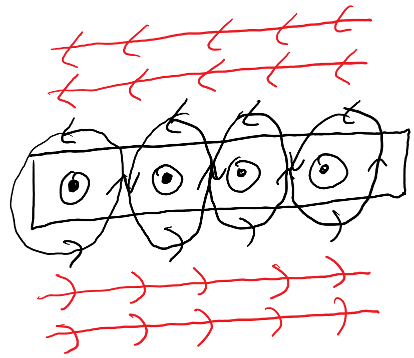
a) Let's find the B field of an infinite plane with surface current $\vec{j}$

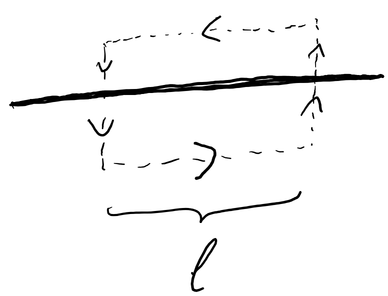
Top view



looking onto plane from front

effective B-field



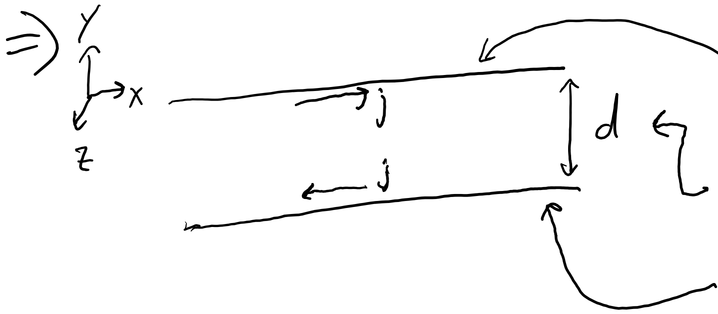


$$\oint \vec{B}\, d\vec{l} = \mu_0 I_{encl.}$$

$$2|B|l = \mu_0 \, j \, l$$

$$\Rightarrow |\vec{B}| = \frac{\mu_0 |\vec{j}|}{2}$$

We can now simply use superposition of the fields (which don't change in strength with distance away from the sheet) coming from the two sheets. On the top and the bottom of the top and the bottom sheet respectively, the fields coming from the two sheets will be equal, but opposite in direction as can be easily seen from the above sketch indicating the direction of the B field. In the middle between the sheets, the fields will point into the same direction and therefore add up, giving us the following expression for the fields:

$\Rightarrow$ 

$$\boxed{\begin{aligned} \vec{B}_{top} &= 0 \\ \vec{B}_{mid} &= -\mu_0 |\vec{j}|\ \hat{z} \\ \vec{B}_{bot.} &= 0 \end{aligned}}$$

b) We know that $dF = dl\ \vec{I} \times \vec{B}$

We have: $\vec{j} \cdot d = I$ (i.e. like if you would have many wires in parallel next to each other)

We also know that dl runs over a distance of a

We also know that $\vec{B} \perp \vec{I}$

$$\Rightarrow F = a \cdot ja \cdot B = B a^2 j = \frac{\mu_0 j^2 a^2}{2}$$

we only take the B field of one of the planes since the plane can't act a force upon itself

using the right hand rule we see that the force points upwards

$$\Rightarrow \text{alltogether we get } \boxed{\vec{F} = \frac{\mu_0 j^2 a^2}{2}\ \hat{y}}$$