Physics 137A: Quantum Mechanics I, Fall 2020

Final Exam

Reminder: you can use Griffiths, the course slides and other materials, and your own notes, but no other sources and no internet or other communication. You can read these materials on a screen but cannot search using the keyboard. A basic calculator is OK but not necessary. Your screen name should be your name or student ID.

The **six** problems each count for 25 points. Submission of your test to bCourses should start by three hours after exam start. Please keep your Zoom camera on and send me a chat message if you need a short break; you do not need to wait for a response from me. I can't really answer questions about the exam since not all students may read my answers, and if there are technical challenges like a bad Internet connection, please still submit your exam at the expected time.

0. **Honor pledge:** At the top of your exam, write "As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others", and sign your name.

1. (a) (6 points) Find a Dirac-normalized eigenstate of the position operator in one dimension with eigenvalue x_0.

(b) (6 points) (i) What is the commutator of the position operator and a potential energy operator $V(x)$?

(ii) Are these eigenstates of the position operator also eigenstates of every potential energy operator $V(x)$?

(iii) If so, what is the eigenvalue of the potential energy operator if the eigenvalue of position is x_0?

(c) (6 points) (i) Write a generalized uncertainty relation for position and kinetic energy $K = p^2/(2m)$.

(ii) In what states is this lower bound minimized?

(d) (7 points) Suppose that a particle in 1D has the momentum-space wavefunction

$$\phi(p) = (\delta(p - p_0) + \delta(p + p_0)). \tag{1}$$

What is the corresponding real-space wavefunction $\psi(x)$?

2. Suppose that we have a two-dimensional Hilbert space with orthonormal basis vectors χ_1 and χ_2. Suppose that Hermitian operator O_1 has these basis vectors as its eigenstates, with different eigenvalues λ_1 and λ_2.

(a) (5 points) (i) Is the state $\psi = \frac{1}{2}\chi_1 + \frac{\sqrt{3}}{2}\chi_2$ normalized? If not, normalize it.

(ii) What are the probabilities to measure λ_1 and λ_2 when the observable described by O_1 is measured in this state?

(b) (4 points) Suppose that the outcome of this measurement is λ_1. A second measurement of O_1 is performed immediately after that outcome was obtained. What are the possible outcomes and their probabilities of this repeated measurement?

(c) (7 points) Suppose that the Hamiltonian of the system, in the χ_1, χ_2 basis, has the form

$$H = \begin{pmatrix} E & E \\ E & E \end{pmatrix}. \tag{2}$$

(i) What are the energy eigenvalues of the system?

(ii) What are the eigenstates, as linear combinations of χ_1 and χ_2?

(d) (6 points) If the system starts in state χ_1 and evolves according to this Hamiltonian, what is its probability that, if O_1 is measured at some later time t, the outcome is λ_1? The time evolution is under the Hamiltonian in (c).

(e) (3 points) Another operator M in the same Hilbert space has eigenvectors χ_1 and $(\chi_1+\chi_2)/\sqrt{2}$, with two distinct eigenvalues a_1 and a_2. Can this operator M represent an observable quantity? Explain why or why not.

3. (a) (5 points) The muon is a particle with the same charge as the electron, but approximately 200 times heavier. Estimate the energy of the 1s orbital of an artificial atom made up of a proton and a muon, in electron volts. An answer within twenty percent is fine.

(b) (10 points) Suppose that we superpose the lowest and first excited states of a 1D harmonic oscillator of classical frequency ω, so that the initial state is $(c_0+c_1x)\exp(-\alpha x^2)$, for some positive constants c_0 and c_1. (i) Is x_0 positive or negative? You do not need to calculate its numerical value if you can explain your answer by other means.

(ii) Compute $\langle x\rangle$ as a function of time, in terms of its initial value $x_0 = \langle x\rangle_{t=0}$.

(c) (5 points) (i) Using the $\ell = 1$ spherical harmonics, make a superposition with $\langle z\rangle > 0$. You may wish to think about (c). You do not need to normalize your answer or do an explicit calculation if you can explain why your answer has $\langle z\rangle > 0$ with a picture.

(ii) How will $\langle z\rangle$ evolve in time? Justify your answer.

(d) (5 points) A neutral atom of neon has 10 electrons. List the $(n,\, \ell,\, m_\ell,\, m_s)$ quantum numbers of the orbitals that these electrons will fill in the ground state.

4. Harmonic oscillators: (a) (5 points) For a particle of mass m moving in a one-dimensional harmonic oscillator $V(x) = kx^2/2$, write the ground state energy and first excited state energy.

(b) (5 points) The 1D ground state wavefunction is proportional to a Gaussian, $\psi_0 = c_0 \exp(-\alpha x^2)$, where c_0 and α are constants depending on m and k. (You do not need to normalize the state or find c_0 and α.)

Use this wavefunction to write the ground state(s) wavefunction of a particle moving *in two dimensions* in the harmonic oscillator potential: $V(x, y) = k(x^2 + y^2)/2$.

What is the energy of the 2D ground state(s)? What is the degeneracy (that is, how many states have this energy)?

(c) (5 points) The first excited state in 1D is proportional to the same Gaussian times x, $\psi_1 = c_1 x \exp(-\alpha x^2)$.. With the help of this state, write the first excited state(s) of the 2D harmonic oscillator.

What is the energy of the 2D first excited state(s)? What is the degeneracy?

(d) (5 points) Is the harmonic oscillator a central potential? What does being a central potential imply about the angular momentum operator L_z and the Hamiltonian? Write an expression for L_z in terms of momentum and position operators.

(e) (5 points) Are the states found in (b) and (c) angular momentum eigenstates (i.e., eigenstates of L_z)? If so, what are the eigenvalues? If not, can you either form angular momentum eigenstates from them and give their eigenvalues, or explain why this can't be done?

5. Consider the 3p electron orbitals ($n = 3$, $\ell = 1$) of the hydrogen atom. Including spin, there are 6 such orbitals: these orbitals form a 6-dimensional Hilbert subspace. So each of your answers below should include a total of 6 states. You do not need to write any explicit forms for the states or give their "quantum numbers" (eigenvalues) other than the ones requested in each question. 3 points except for (g), 4 points for (g).

(a) What are the possible eigenvalues of L_z (the z-component of orbital angular momentum), and their degeneracies (the number of states with each possible value)?

(b) What are the possible eigenvalues of L^2, and their degeneracies?

(c) What are the possible eigenvalues of S_z, and their degeneracies?

(d) What are the possible eigenvalues of S^2, and their degeneracies?

(e) What are the possible eigenvalues of $L_z + S_z$ and their degeneracies?

(f) What are the possible eigenvalues of $(\mathbf{L}+\mathbf{S})^2$ and their degeneracies?

(g) What are the possible eigenvalues of $\mathbf{L}\cdot\mathbf{S}$ and their degeneracies?

(h) What are the possible eigenvalues of $L_y + S_y$ and their degeneracies? (yes, this is meant to be y and not z)

6. Consider the infinite square well in one dimension, with infinite potential walls at $x = 0$ and $x = a$.

(a) (5 points) Which is larger, the ground state energy of one well with a particle of mass $2m$, or twice the ground state energy of one well with a particle of mass m?

(b) (10 points) Suppose that the initial state of a particle is uniform in the box: $\psi(x) = c_0$. (i) What must the constant c_0 be by normalization?

(ii) *Assume* that this wavefunction can be expanded over energy eigenstates. What is the probability P_n to observe energy E_n in an energy measurement?

(iii) Is the expectation value of energy finite?

(c) (10 points) The point of this part is to figure out the energy eigenvalues without doing any hard calculations. Suppose that we added a delta-function potential in the middle of the well,

$$V' = \alpha\delta(x - a/2). \tag{3}$$

with $\alpha > 0$. (i) If α is very large, what are the energy eigenvalues?

Hint: are some of the energy eigenvalues unmodified?

For the others, take advantage of the fact that $\alpha \to \infty$ to relate the problem to one that you know how to solve.

(ii) What is the ground state energy in this limit? Is it degenerate?

(iii) Write a criterion for α to be "large" (that is, large compared to what?) and justify it.