Physics 137A: Quantum Mechanics I, Fall 2020

Midterm II, November 19, 2020

Reminder: you can use Griffiths, the course slides and other materials, and your own notes, but no other sources and no internet or other communication. A basic calculator is OK but not necessary. The four problems count equally and increase in difficulty, more or less. Submission of your test to bCourses should start by 11 am except by prior arrangement. I cannot answer questions during the exam.

1. (a) $\hbar/(30 \text{ minutes})$. Almost full credit given for 15 minutes in the denominator.

(b) Solving the differential equation and imposing the periodic boundary condition leads to $e^{im\phi}, m \in \mathbb{Z}$.

(c) $\ell = 0, 1, 2, \ldots$, $m = -\ell, -\ell + 1, \ldots, \ell$. $Y_{\ell m}$ is even when ℓ is even.

(d) Yes in $d = 1$ and no in $d = 3$, as we might conclude from the finite square well in both cases.

2. (a) Normalization is that the inner product of the state with itself should give 1, which implies $|a|^2 + |b|^2 = 1$. The time evolution is, reading off the energy eigenvalues from this diagonal matrix,

$$a e^{-iE_0 t/\hbar} |e_1\rangle + b e^{iE_0 t/\hbar} |e_2\rangle \tag{1}$$

(b) There are two possible values, E_0 with probability $|a|^2$ and $-E_0$ with probability $|b|^2$. That gives mean energy $\langle H \rangle = E_0(|a|^2 - |b|^2)$ and variance $\langle H^2 \rangle - \langle H \rangle^2 = E_0^2(1 - (|a|^2 - |b|^2)^2)$. As expected, the variance vanishes if the system is in either energy eigenstate.

(c) Yes, it has to have the same eigenvectors. No, it does not have to have the same eigenvalues.

(d) Yes, it is Hermitian and its matrix is

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

(e) The product of these two projection operators is just 0, so the result is independent of the initial state.

3. (a) The uncertainty relation gives

$$\sigma_{p_x} \geq \frac{\hbar}{2\sigma_x}. \tag{3}$$

Then $\langle {p_x}^2 \rangle$ must be at least $\sigma_{p_x}^2$ by the definition of variance. Spherical symmetry implies the same for the y and z components. So the kinetic energy is at least $\frac{3}{2m}\frac{\hbar^2}{4\sigma_x^2}$.

(b) The total potential energy is at least $(3k/2)\sigma_x^2$.

(c) There is a minimum at a positive value of σ_x, since the kinetic part diverges when the uncertainty is small, and potential energy diverges when uncertainty is large. We can find the minimum by setting the first derivative of the total energy to zero. That gives

$$\sigma_x = \left(\frac{\hbar^2}{4mk}\right)^{1/4}. \tag{4}$$

At this value, kinetic and potential energies are both $3\hbar\omega/4$, with $\omega = \sqrt{k/m}$, and the total energy is $3\hbar\omega/2$. This is the actual ground state energy, and the agreement makes sense once we recall that the Gaussian ground state of the harmonic oscillator realizes the minimum of the uncertainty relation.

4. (a)

$$c(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar}\, dx. \tag{5}$$

(b)

$$\int_{p_a}^{p_b} |c(p)|^2\, dp. \tag{6}$$

(c)

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x)(-\hbar^2)\frac{\partial^2\psi}{\partial x^2}\, dx = \int_{-\infty}^{\infty} |c(p)|^2 p^2\, dp. \tag{7}$$

(d) Doing the integral gives $\frac{\sqrt{a}}{\sqrt{\hbar}} e^{-a|p|/\hbar}$, and a check is that the integral of $|c(p)|^2$ gives 1, as it should by normalization.