Physics 137A: Quantum Mechanics I, Fall 2020

Midterm, October 20, 2020

Reminder: you can use Griffiths, the course slides and other materials, and your own notes, but no other sources and no internet or other communication. A basic calculator is OK but not necessary. The four problems count equally and increase in difficulty, more or less. Submission of your test to bCourses should start by 11 am except by prior arrangement. I cannot answer questions during the exam; I will try to find a way to do that for the final.

1. At $t = 0$, suppose that a state in the infinite square well with potential

$$V(x) = \begin{cases} \infty & \text{if } x < 0 \text{ or } x > a \\ 0 & \text{if } 0 \leq x \leq a \end{cases} \tag{1}$$

has the wavefunction

$$\psi(x, t = 0) = \frac{1}{\sqrt{2}}\psi_2(x) + \frac{1}{\sqrt{3}}\psi_3(x) + \frac{1}{\sqrt{6}}\psi_4(x), \tag{2}$$

where the normalized eigenstates are chosen to be $\psi_n = \sqrt{\frac{2}{a}} \sin(n\pi x/a)$, $n = 1, 2, 3, \ldots$. The Hamiltonian is

$$H = \frac{p^2}{2m} + V(x). \tag{3}$$

(a) (5 points) What are the eigenvalues of energy in this potential?

(b) (5 points) Give the possible values seen in a measurement of energy on ψ from (2), and the probabilities to see them.

(c) (5 points) What is the expectation value of energy in this state?

(d) (5 points) Find the wavefunction $\psi(x, t)$ at all later times $t > 0$.

(e) (5 points) What is the probability distribution of an energy measurement in this state at arbitrary times t?

2. (a) (5 points) Suppose we take a *free* particle's initial state to be the (normalized) ground state of the infinite square well from Problem 1. Is this state an energy eigenstate of the free particle Hamiltonian?

(b) (20 points) Let's work with the shifted state $\psi(x) = \psi_1(x + a/2)$, which is even about the origin. Sketch this state. Write an expansion (Fourier transform) for this state, in terms of the free particle eigenstates e^{ikx}. For full credit, do the integral to find $\phi(k)$.

3. (a) (13 points) Consider the bound state of the δ-function potential, with wavefunction proportional to $e^{-\kappa|x|}$.

Possibly useful integral: $\int_0^\infty x^2 e^{-\alpha x}\, dx = 2/\alpha^3$.

Find σ_x in terms of κ and then use the uncertainty principle to state a lower bound on the momentum uncertainty σ_p.

(b) (12 points) Consider unbound states of particles moving in one dimension where the potential $V(x)$ goes to zero rapidly (say, exponentially) as $x \to \pm\infty$.

Answer the following questions with "yes" or "no" and give an example to support your reasoning. 3 points each.

(b1) If $V(x)$ diverges at a point, can the transmission coefficient $T(E)$ still be nonzero for a finite energy E?

(b2) If $V(x)$ is nonzero in some region, can the transmission probability $T(E)$ be exactly one for a finite energy E?

(b3) With the above limits for the potential at spatial infinity, can there be unbound states at $E < 0$?

(b4) Is it possible for $V(x)$ to have the same $T(E)$ for all positive energies as $-V(x)$?

4. Consider the harmonic oscillator with ground state wavefunction $\psi_0(x) = Ae^{-m\omega x^2/2\hbar}$, with $A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$, which makes this a normalized wavefunction.

For this problem, you may wish to use the Gaussian integrals

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}, \quad \int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \sqrt{\frac{\pi}{4a^3}}. \tag{4}$$

However, in most of these cases doing integrals is not the most direct way to solve the problem.

(a) (12 points) Find the normalization constant a_0 in the state $\psi'(x) = a_0 x^2 \psi(x)$, and the distribution of energies in a measurement of energy in this state. Is it an energy eigenstate?

(b) (7 points) Consider the following wavefunction (which is discontinuous at the origin, but that's OK as it still has an expansion over eigenstates)

$$\phi(x) = \begin{cases} c_0\psi_0(x) & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases} \tag{5}$$

Find the constant c_0 from normalization. What is the probability, in an energy measurement on this state, to measure the value $E_0 = \hbar\omega/2$

(c) (6 points) For the state $\phi(x)$ in (c), what is the total probability to measure a value E_n for all **even** values of n, i.e., $\sum_{n \text{ even}} P_n$, where P_n is the probability to measure E_n, including the ground state? Explain your answer.