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Math54 Midterm II (version 1), Fall 2020

This is an open book exam. You are allowed to cite any results, up to Section 1.4 but excluding those in the exercises, from the textbook. Results from anywhere else will be treated the same as your answers, both of which need to be justified. Completely correct answers given without justification will receive little credit. Partial solutions will get partial credit.

Problem	Maximum Score	Your Score
1	2	
2	14	
3	14	
4	14	
5	14	
6	14	
7	14	
8	14	
Total	100	

1. By signing below, you promise you completed the midterm exam work all by yourself. Exams without this signature will not be graded. (If you wish, you are allowed to sign on paper on which the above words in small caps are hand written.)

   Your Name: ______________________________

   Your SID: ______________________________

   Your Section: ______________________________

   Your Signature: ______________________________

2. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

- Verify that $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
- In general, for any integer $k \geq 2$, explain that $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

Solution: Use induction, we assume $A^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ for $k \geq 2$. The case $k = 2$ can be directly verified.

$$A^{k+1} = A^k\,A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}.$$

Thus the equation holds for all positive integer k.

Alternatively, Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^k = \mathbf{0}$ for all $k \geq 2$, it follows that

$$A^k = \left(I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)^k = I + k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

3. Consider the following map from $\mathbf{R}^3$ to $\mathbf{P}_2$:

$$\mathbf{T}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = (\alpha+\beta) + (\beta+\gamma)\,x + (\gamma+\alpha)\,x^2.$$

Is $\mathbf{T}$ a linear transformation? If so, is it an isomorphism?

SOLUTION: $\mathbf{T}$ is a linear transformation because it satisfies

$$\mathbf{T}(\mathbf{x}+\mathbf{y}) = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$$

and

$$\mathbf{T}(c\,\mathbf{x}) = c\,\mathbf{T}(\mathbf{x})$$

for all scalar c and all vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$. We claim $\mathbf{T}$ is one-to-one and onto by showing that for any polynomial $a + b\,x + c\,x^2 \in \mathbf{P}_2$, there exists one and only one vector $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathcal{R}^3$ such that

$$\mathbf{T}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = a + b\,x + c\,x^2.$$

Indeed, this equation is equivalent to

$$\begin{pmatrix} \alpha+\beta \\ \beta+\gamma \\ \gamma+\alpha \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

which is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

or
$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Since there is always a solution, $\mathbf{T}$ must be onto; since there is always a unique solution, $\mathbf{T}$ must be one-to-one.

4. Let matrix B be defined by

$$B = \begin{pmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Solve the least squares problem

$$\mathbf{min}_{\mathbf{x}} \left\| B\,\mathbf{x} - \mathbf{b} \right\|.$$

SOLUTION: Solution is

$$\begin{aligned} \mathbf{x} &= \left(B^T B\right)^{-1} \left(B^T B\right)^{-1} \\ &= \begin{pmatrix} 14 & 19 \\ 19 & 26 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ 18 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -4 \\ 5 \end{pmatrix} \end{aligned}$$

5. Let matrix A be defined by

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ -2 & 0 & 1 \end{pmatrix}.$$

- Compute the row echelon form of A. SOLUTION: the row echelon form is $\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -7 \end{pmatrix}$.
- Calculate $\det(A)$. SOLUTION: $\det(A) = -7$.

6. Let matrix A be defined by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find an orthonormal basis for $\mathbf{Col}\, A$ and an orthonormal basis for $\mathbf{Null}\, A$.

SOLUTION: We do Gram-Schmidt:

- From first column of A, choose $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$
- From second column of A, Choose

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}^T \mathbf{v}_1}{\mathbf{v}_1^T \mathbf{v}_1} \mathbf{v}_1 = \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

- Since $det(A) = 0$, $\mathbf{Col}\, A$ has dimension 2. Normalizing to obtain orthonormal basis

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{30}} \begin{pmatrix} -2 \\ 1 \\ 5 \end{pmatrix}$$

- Row echelon form for A is

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

implying a null space of dimension 1 (one free variable column), with null vector $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, which is normalized to $\frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

7. Let matrix A be defined by

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -2 \end{pmatrix}.$$

Is A diagonalizable? Explain.

SOLUTION:

$$\begin{aligned} \mathbf{det}\,(A - \lambda I) &= \mathbf{det} \begin{pmatrix} 1-\lambda & -2 & 0 \\ 0 & 1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{pmatrix} \\ &= \mathbf{det} \begin{pmatrix} & -(1+\lambda) & -(1-\lambda)\,(2+\lambda) \\ 0 & 1-\lambda & 2 \\ -1 & 1 & -2-\lambda \end{pmatrix} = \lambda\left(\sqrt{5}+\lambda\right)\left(\sqrt{5}-\lambda\right) \end{aligned}$$

So the eigenvalues of A are $0, \pm\sqrt{5}$, and therefore eigenvalues are all distinct. Thus A is diagonalizable.

8. Let matrices $B \in \mathbf{R}^{n\times k}$ and $C \in \mathbf{R}^{k\times n}$ for $0 < k < n$. Define $A = B\,C$. Explain that $\mathbf{rank}(A) \leq k$.

   SOLUTION: Any vector in $\mathbf{Col}\,A$ is of the form $A\,\mathbf{x}$ for some $\mathbf{x} \in \mathcal{R}^n$. Let $\mathbf{b}_1, \cdots, \mathbf{b}_k$ be columns of B, then
$$A\,\mathbf{x} = (\mathbf{b}_1, \cdots, \mathbf{b}_k)\,(C\,\mathbf{x})$$
   is a linear combination of vectors $\mathbf{b}_1, \cdots, \mathbf{b}_k$, therefore the dimension of $\mathbf{Col}\,A$ is at most k. In other words, $\mathbf{rank}(A) \leq k$.