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Math54 Midterm I (version 1), Fall 2020

This is an open book exam. You are allowed to cite any results, up to Section 1.4 but excluding those in the exercises, from the textbook. Results from anywhere else will be treated the same as your answers, both of which need to be justified. Completely correct answers given without justification will receive little credit. Partial solutions will get partial credit.

Problem	Maximum Score	Your Score
1	2	
2	14	
3	14	
4	14	
5	14	
6	14	
7	14	
8	14	
Total	100	

1. By signing below, you promise you completed the midterm exam work all by yourself. Exams without this signature will not be graded. (If you wish, you are allowed to sign on paper on which the above words in small caps are hand written.)

   Your Name: ______________________________

   Your SID: ______________________________

   Your Section: ______________________________

   Your Signature: ______________________________

2. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find all solutions to $A\mathbf{x} = \mathbf{b}$ or explain why there are no solutions.

SOLUTION: Compute the row echelon form of the augmented matrix

$$\left(A \mid \mathbf{b} \right) = \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 4 & 1 & 1 \end{array}\right) \overset{\text{echelon}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -\frac{2}{3} \end{array}\right)$$

The number of pivots in A is 2, whereas the number of pivots in the augmented matrix is $3 > 2$. Thus there are no solutions in the linear equations.

3. Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} \in \mathcal{R}^3.$$

Find conditions on $\mathbf{b}$ under which $A\,\mathbf{x} = \mathbf{b}$ has at least one solution.

SOLUTION: Compute the row echelon form of the augmented matrix

$$\left(\, A \mid \mathbf{b} \, \right) = \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 1 & b_2 \\ 4 & 1 & b_3 \end{array} \right) \overset{\text{echelon}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -3 & b_2 - 2\,b_1 \\ 0 & 0 & \frac{1}{3}\,(3\,b_3 + 2\,b_2 - 7\,b_1) \end{array} \right)$$

The number of pivots in A is 2, whereas the number of pivots in the augmented matrix is 2 if and only if $3\,b_3 + 2\,b_2 - 7\,b_1 = 0$. Thus there is at least one solution in the linear equations if and only if $3\,b_3 + 2\,b_2 - 7\,b_1 = 0$.

4. Let set

$$\mathcal{S} = \left\{ \begin{pmatrix} 1+\alpha \\ -\alpha \\ 2\,\alpha \end{pmatrix} \quad \middle| \quad \alpha \text{ is a real scaler.} \right\}$$

Is $\mathcal{S}$ a vector space? Justify your answer.

SOLUTION: It is clear that there does not exist any α such that

$$\begin{pmatrix} 1+\alpha \\ -\alpha \\ 2\,\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore, the zero vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is not in $\mathcal{S}$ and therefore $\mathcal{S}$ can not be a subspace.

5. Let

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 & -1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Find $\mathbf{det}\,(B)$.

SOLUTION:

$$\begin{aligned} \mathbf{det}\,(B) &= -\mathbf{det}\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 3 & -1 & -3 \\ 0 & 1 & 1 & 1 \end{pmatrix} = -\mathbf{det}\begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \\ &= 2\,\mathbf{det}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \mathbf{det}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -1 \end{aligned}$$

6. Let $A \in \mathcal{R}^{3\times 4}$ and $B \in \mathcal{R}^{4\times 3}$. Assume that there are exactly 2 pivots in the row echelon form of A. Explain why $\mathbf{det}\,(A\,B) = 0$.

   Solution: Let U be the row echelon form for A, so that there are a number of elementary matrices $E_1, \cdots, E_p$ such that

$$E_1 \cdot \cdots \cdot E_p\, A = U \quad \text{and} \quad A = (E_1 \cdot \cdots \cdot E_p)^{-1}\, U.$$

   Since there are exactly 2 pivots in the row echelon form of A, the last row of U must be 0. Therefore the last row of $U\,B$ must also be 0 so that $\mathbf{det}\,(U\,B) = 0$. Hence

$$\mathbf{det}\,(A\,B) = \mathbf{det}\left((E_1 \cdot \cdots \cdot E_p)^{-1}\right)\,\mathbf{det}\,(U\,B) = 0.$$

7. Let

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

Calculate $A^\top A$ and $A A^\top$.

SOLUTION:

$$\begin{aligned}
\mathbf{det}\left(A^\top A\right) &= \mathbf{det}\left(A^\top\right)\mathbf{det}\,(A) = (\mathbf{det}\,(A))^2 = 5^2 = 25 \\
\mathbf{det}\left(A\,A^\top\right) &= \mathbf{det}\,(A)\,\mathbf{det}\left(A^\top\right) = (\mathbf{det}\,(A))^2 = 5^2 = 25
\end{aligned}$$

8. Consider the matrix transformation

$$\mathbf{x} \Longrightarrow A\,\mathbf{x} \in \mathcal{R}^3, \quad \text{for all} \quad \mathbf{x} \in \mathcal{R}^2$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

- Is the transformation one-to-one?
- Is the transformation onto?

SOLUTION:

- Let $A\,\mathbf{x}_1 = A\,\mathbf{x}_2$ for $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{R}^2$. Then $A\,(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$. Since columns of A are linearly independent, it follows that $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$ or $\mathbf{x}_1 = \mathbf{x}_2$. Therefore transformation is one-to-one
- Consider the vector $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. We want to show that there is no $\mathbf{x}$ such that $A\,\mathbf{x} = \mathbf{b}$. Indeed, the augmented matrix

$$\left(A \,\middle|\, \mathbf{b} \right) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & -2 \\ 1 & 3 & 1 \end{array}\right) \overset{\text{echelon}}{\Longrightarrow} \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{array}\right)$$

has one more pivot than A, leading to no solution in $A\,\mathbf{x} = \mathbf{b}$. Thus transformation can not be onto.