IEOR 165 – Midterm Exam Solutions
Spring 2021

Instructions:

- Open notes/homeworks/solutions only
- Calculators are allowed (graphing calculators are okay)
- Excel/R/Python/similar softwares are **not** allowed
- If steps/works are not shown, then points will be deducted
- Communicating with anyone other than GSI/instructor is **not** allowed
- Exam period **starts** Wednesday, March 17 at 2PM (Pacific Time) and **ends** Thursday, March 18 at 2PM (Pacific Time)
- Once you start the exam, you will have **1.5 hours** to complete the exam
- There are extra office hours during which you can get clarification about the exam questions

Name: ______________________________

Student ID: ______________________________

1	/10
2	/10
3	/10
4	/10
5	/10

1. We are growing beans in science class and would like to study the effect of fertilizer amount (F) on the height of the bean stalks (H) after a month of time. The experiment was performed on three bean stalks with different values of F. The data collected is shown below.

H	F
1.4	10
2.2	15
5.0	30
6.5	45
8.0	50
8.2	60

Consider the linear model $H_i = \beta F_i + \epsilon_i$ where i denotes the i^{th} data. Assume that the noises $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ are iid with known σ^2, and that the prior distribution of the coefficient is Gaussian: $\beta \sim \mathcal{N}(0, 1/(2\lambda))$. Suppose $\sigma^2 = 0.3$ and $\lambda = 2$. Find the estimate for β using the Maximum A Posteriori (MAP) method. (10 points)

Solution: The MAP estimator is the solution to the following optimization problem:

$$\arg\max_{\beta} \ \exp\left(\sum_{i=1}^{6} -(H_i - \beta F_i)^2/(2\sigma^2)\right) \cdot \exp\left(-\lambda\beta^2\right) \tag{1}$$

Taking the log and negating the problem, we arrive at an equivalent minimization problem:

$$\arg\min_{\beta} \ \sum_{i=1}^{6} (H_i - \beta F_i)^2 + 2\lambda\sigma^2\beta^2 \tag{2}$$

Take the derivative with respect to β and set it to zero.

$$\left(\sum_{i=1}^{6} 2F_i^2 + 4\lambda\sigma^2\right)\beta - \sum_{i=1}^{6} 2H_iF_i = 0 \tag{3}$$

Solving for β we get:

$$\hat{\beta} = \left(\sum_{i=1}^{6} H_iF_i\right) \Big/ \left(\sum_{i=1}^{6} F_i^2 + 2\lambda\sigma^2\right) = 0.1477 \tag{4}$$

2. Suppose that we are flipping a 'tilted' coin that shows a head with probability p and a tail with probability $1 - p$. Let X be a random variable that counts the number of heads after flipping the coin n times. Then, $X \sim Binomial(n, p)$ with mean np and variance $np(1-p)$. After repeating the experiment five times (each experiment consists of n coin flips), we observe $X_1, \ldots, X_5$ to be equal to $15, 20, 18, 22, 20$. Use the method of moments to estimate the parameters n and p. (10 points)

Solution: Let us first express the first and second moments as a function of the distribution parameters.

$$\mu_1 = \mathbb{E}[X] = np \tag{5}$$

$$\mu_2 = \mathbb{E}[X^2] = \sigma^2 + \mu_1^2 = np(1-p) + n^2p^2 \tag{6}$$

To invert these functions, note that substituting the equation for μ_1 into the equation for μ_2 gives

$$\mu_2 = \mu_1(1-p) + \mu_1^2 \quad \Rightarrow \quad p = \frac{\mu_1 - \mu_2 + \mu_1^2}{\mu_1} \tag{7}$$

Substituting this equation for μ_1 gives

$$\mu_1 = n \cdot \frac{\mu_1 - \mu_2 + \mu_1^2}{\mu_1} \quad \Rightarrow \quad n = \frac{\mu_1^2}{\mu_1 - \mu_2 + \mu_1^2} \tag{8}$$

An estimator of the first and second moments are

$$\hat{\mu_1} = \frac{1}{5}\sum_{i=1}^{5} X_i = \frac{1}{5}(15 + 20 + 18 + 22 + 20) = 19 \tag{9}$$

$$\hat{\mu_2} = \frac{1}{5}\sum_{i=1}^{5} X_i^2 = \frac{1}{5}(15^2 + 20^2 + 18^2 + 22^2 + 20^2) = 366.6 \tag{10}$$

Plugging these values into the equations for n and p, we get

$$\hat{n} = \frac{19^2}{19 - 366.6 + 19^2} = 26.94 \simeq 27 \tag{11}$$

$$\hat{p} = \frac{19 - 366.6 + 19^2}{19} = 0.7053 \simeq 0.7 \tag{12}$$

3. Let $a > 0$, $\theta > 0$ and $X_1, \ldots, X_n$ be iid random variables from the probability distribution function

$$f_{a;\theta}(x) = \frac{a-1}{\theta^{a-1}} \cdot x^{a-2}, \quad 0 < x < \theta$$

The first and second moments of the above distribution is

$$\mathbb{E}[X] = \frac{a-1}{a}\theta, \quad \mathbb{E}[X^2] = \frac{(a-1)\cdot\theta^2}{a+1} \tag{13}$$

Use the method of moments to derive an estimator of a and θ. (10 points)

Solution: Let $\mu_1 = \mathbb{E}[X]$, $\mu_2 = \mathbb{E}[X^2]$. To invert the functions, note that substituting the equation for μ_1 into the equation for μ_2 gives

$$\mu_2 = \frac{a-1}{a+1}\left(\frac{a}{a-1}\right)^2 \mu_1^2 \tag{14}$$

Solving this equation for a, we get

$$a = \sqrt{\frac{\mu_2}{\mu_2 - \mu_1^2}} \tag{15}$$

$$\theta = \frac{a}{a-1}\mu_1 = \frac{\mu_1 \cdot \sqrt{\frac{\mu_2}{\mu_2-\mu_1^2}}}{\sqrt{\frac{\mu_2}{\mu_2-\mu_1^2}} - 1}. \tag{16}$$

Finally, simply plug in the sample estimates for μ_1 and μ_2,

$$\hat{\mu_1} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{17}$$

$$\hat{\mu_2} = \frac{1}{n}\sum_{i=1}^{n} X_i^2. \tag{18}$$

$$\hat{a} = \sqrt{\frac{\hat{\mu}_2}{\hat{\mu}_2 - \hat{\mu}_1^2}}, \quad \hat{\theta} = \frac{\hat{\mu}_1 \cdot \sqrt{\frac{\hat{\mu}_2}{\hat{\mu}_2-\hat{\mu}_1^2}}}{\sqrt{\frac{\hat{\mu}_2}{\hat{\mu}_2-\hat{\mu}_1^2}} - 1} \tag{19}$$

4. (a) Suppose $x_1, \ldots, x_n$ are iid samples from a distribution with density

$$f_\theta(x) = \frac{\theta}{x^2}, \quad 0 < \theta \le x$$

Find the maximum likelihood estimator (MLE) of θ. (5 points)

(b) Suppose $y_1, \ldots, y_n$ are iid samples from a distribution with density

$$f_\theta(y) = \frac{2}{\theta} \cdot y \cdot \exp\{-\frac{y^2}{\theta}\}, \quad y > 0, \quad \theta > 0$$

Find the maximum likelihood estimator (MLE) of θ. (5 points)

Solution:

(a)

$$L(\theta) = \begin{cases} \prod_{i=1}^n \frac{\theta}{x_i^2}, & 0 < \theta \le x_i \text{ for all } i \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{\theta^n}{\prod_{i=1}^n x_i^2}, & 0 < \theta \le \min\{x_1, \ldots, x_n\} \\ 0, & \text{otherwise} \end{cases}$$

$$\ln L(\theta) = n \ln \theta - 2 \sum_{i=1}^n \ln x_i$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta} > 0$$

Since the log-likelihood is strictly increasing in θ (where the likelihood is not zero), the MLE is the largest allowable value of θ in this region, i.e., $\hat{\theta}_{\text{MLE}} = \min\{x_i\}$.

(b)

$$L(\theta) = (\frac{2}{\theta})^n (\prod_{i=1}^n y_i) \exp\{-\frac{1}{\theta} \sum_{i=1}^n y_i^2\}$$

$$\ln L(\theta) = n(\ln 2 - \ln \theta) + \sum_{i=1}^n \ln y_i - \frac{1}{\theta} \sum_{i=1}^n y_i^2$$

$$\frac{d}{d\theta} \ln L(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^n y_i^2$$

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^n y_i^2$$

5. We propose the following model for the time it takes to perform a simple task as a function of the number of times the task has been practiced

$$T \approx ts^{-n}$$

where T is the time, n is the number of times the task has been practiced, and t and s are parameters depending on the task and individual. Given the following data set

T	22.4	21.3	19.7	15.6	15.2	13.9
n	0	1	2	3	4	5

(a) Estimate t and s. (7 points)

(b) Estimate the time it takes to perform the task after 6 practices. (3 points)

Solution: Using least squares,

$$\begin{aligned}
\log T &= \log t - n \log s = \log t - \log s \cdot n \\
\widehat{\log t} &= 3.1342, \ \widehat{\log s} = 0.1038 \\
\hat{t} &= 22.970, \ \hat{s} = 1.109 \\
\hat{T}(6) &= 22.970 \cdot (1.109)^{-6} = 12.325
\end{aligned}$$