EECS 16A Designing Information Devices and Systems I
Fall 2020 Final Exam – SOLUTIONS

1. Save Baby Yoda! (8 points)

Despite our best efforts, we have lost Baby Yoda to former agents of the Galactic Empire. Luckily we were able to conceal a receiver in his locket, so now it's time to save Baby Yoda using our 16A knowledge!

Baby Yoda has been delivered to an Imperial Star Destroyer. Rebel intel has provided us with access to their internal communication beacons. The ship's layout is 2-dimensional with 3 beacon locations specified in Table 1.

Beacon	Coordinates	Distance to Baby Yoda
A	(5,5)	$\sqrt{20}$
B	(2,3)	1
C	(1,1)	2

Table 1: Data from Destroyer Beacons and their coordinates.

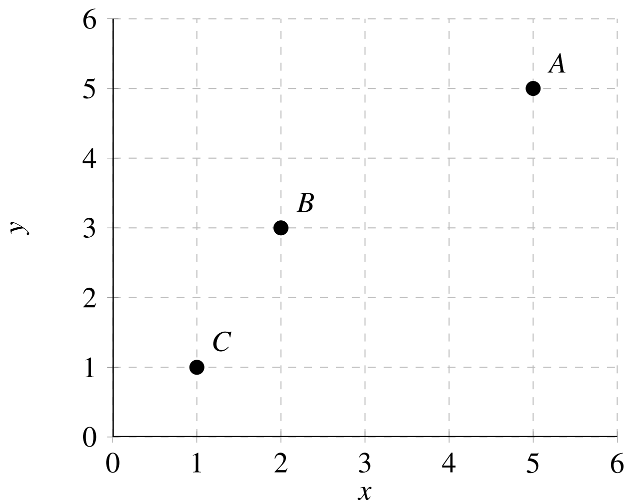


Figure 1: Diagram of the Destroyer's floor-plan with Beacon coordinates marked accordingly.

Explicitly write out a *linear* system of equations (in matrix-vector form) using the data above for finding Baby Yoda's location $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Draw a box around your final linear system, then solve for Baby Yoda's location. Nonlinear terms are not permitted in your final system of equations. You must provide both the system and the location for full credit.

Solutions:

The three equations are:

$$
\begin{aligned}
[A]: \quad (x-5)^2+(y-5)^2 &= x^2 - 10x + 25 + y^2 - 10y + 25 = 20, \\
[B]: \quad (x-2)^2+(y-3)^2 &= x^2 - 4x + 4 + y^2 - 6y + 9 = 1, \\
[C]: \quad (x-1)^2+(y-1)^2 &= x^2 - 2x + 1 + y^2 - 2y + 1 = 4.
\end{aligned}
$$

Method A: We subtract equation [A] from the others to yield:

$$\begin{aligned} 6x - 37 + 4y &= -19 \\ 8x - 48 + 8y &= -16 \end{aligned} \quad\longrightarrow\quad \begin{aligned} 3x + 2y &= 9 \\ x + y &= 4 \end{aligned} \quad\longrightarrow\quad \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}.$$

Method B: We subtract equation [B] from the others to yield:

$$\begin{aligned} -6x + 37 - 4y &= 19 \\ 2x - 11 + 4y &= 3 \end{aligned} \quad\longrightarrow\quad \begin{aligned} 3x + 2y &= 9 \\ x + 2y &= 7 \end{aligned} \quad\longrightarrow\quad \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \end{bmatrix}.$$

Method C: We subtract equation [C] from the others to yield:

$$\begin{aligned} -8x + 48 - 8y &= 16 \\ -2x - 11 - 4y &= -3 \end{aligned} \quad\longrightarrow\quad \begin{aligned} x + y &= 4 \\ x + 2y &= 7 \end{aligned} \quad\longrightarrow\quad \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

All of these systems of equations are acceptable. In the following solution we will work from the Method A system. From this stage there are three ways to solve for Baby Yoda's location:

(1) Row Reduction: Use Gaussian elimination on your system.

$$\left[\begin{array}{cc|c} 3 & 2 & 9 \\ 1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 9 \\ 2\cdot 1 - 3 & 2\cdot 1 - 2 & 2\cdot 4 - 9 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 3 & 2 & 9 \\ 1 & 0 & 1 \end{array}\right]$$

Thus $x = 1$. From here we can substitute into either equation: $x + y = 4 \rightarrow y = 3$.

(2) Inverse: Compute the inverse matrix, as per our formula for 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

From this point it becomes a matter of matrix vector multiplication to identify $\vec{x}$ using the relation $\mathbf{A}\vec{x} = \vec{b} \rightarrow \vec{x} = \mathbf{A}^{-1}\vec{b}$.

$$\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \longrightarrow \vec{x} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 9-8 \\ -3+12 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(3) Graphically: Using Figure 1, draw circles centered at each beacon with the appropriate radius from the beacon data. Their intersection identifies Baby Yoda's location.

The solution for Baby Yoda's location is $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

2. Ultrasound Sensing with Op-Amps (14 points)

The transresistance amplifier is often used to convert a current from a sensor to a voltage. In this problem we will use it to build an ultrasound sensor! When an ultrasonic wave hits our sensor, it generates a current, i_{ultra}. Whenever no ultrasonic wave hits our sensor zero current is generated, so $i_{\text{ultra}} = 0$.
*Note: An **ideal op-amp** is used in all subparts of this question. You can also assume that $V_{DD} = -V_{SS}$.*

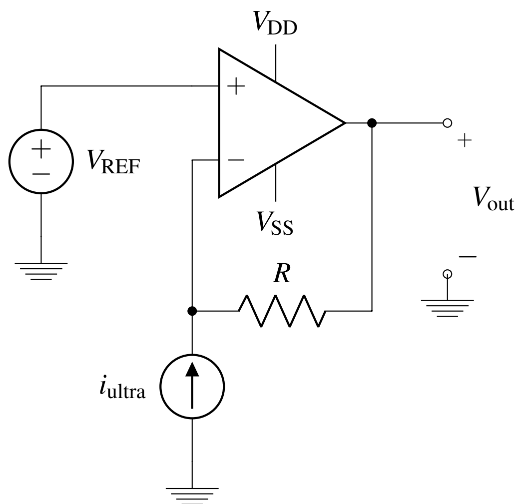


Figure 2: Transresistance sensor circuit

(a) (4 points) **Calculate the output voltage, V_{out}, of the transresistance sensor circuit shown in Fig. 2, as a function of the reference voltage, V_{REF}, the sensor input current, i_{ultra}, and the resistor, R, when an ultrasonic wave hits the sensor.** Clearly show all your work and justify your answer. Writing only the final expression will not be given full credit.

Solutions:
Since we have an ideal op-amp in a negative feedback circuit, we can first say that $u_- = u_+ = V_{\text{REF}}$. Next the current i_{ultra} must flow right though R due to KCL and the golden rules (no current can flow into the op-amp). Thus we establish

$$V_{\text{out}} = u_- - i_{\text{ultra}} R = u_+ - i_{\text{ultra}} R = V_{\text{REF}} - i_{\text{ultra}} R. \qquad \square$$

(b) (5 points) Assume that the amplitude of the ultrasonic wave hitting the sensor is such that the current i_{ultra} fluctuates from a minimum value of $i_{\text{min}} = 1 \cdot 10^{-6}$A, to a maximum value of $i_{\text{max}} = 2 \cdot 10^{-6}$A. Also assume that the reference voltage is set to $V_{\text{REF}} = 1$V. In this case, calculate the following:

i. **The maximum value of the resistor, R, so that the output voltage, V_{out}, does not drop below 0V.** Clearly show all your work.

ii. Assuming you picked $R = 250 \cdot 10^3 \Omega$ (which may or may not be the correct answer to part (i)), **calculate the maximum value of the output voltage, V_{out}.** Clearly show all your work.

Solutions:

From part (a) we identified the output voltage of our sensor circuit $V_{\text{out}} = V_{\text{REF}} - i_{\text{ultra}} R$.

i. The worst-case scenario (in which the output voltage is most reduced) occurs for $i_{\text{ultra}} = i_{\max} = 2 \cdot 10^{-6}$A. If we include $V_{\text{REF}} = 1$V and set our nonnegative condition on V_{out}, we identify the restriction on R:

$$V_{\text{out}} = V_{\text{REF}} - i_{\max} R \geq 0 \quad \longrightarrow \quad R \leq \frac{V_{\text{REF}}}{i_{\max}} = \frac{1V}{2 \cdot 10^{-6} A} = 500,000\ \Omega. \qquad \square$$

ii. Based on our voltage formula, the highest V_{out} scenario now occurs for the low current condition $i_{\text{ultra}} = i_{\min} = 1 \cdot 10^{-6}$A. From this point we now substitute into the V_{out} formula:

$$V_{\text{out}} = V_{\text{REF}} - i_{\min} (250 \cdot 10^3 \Omega) = 1 - (1 \cdot 10^{-6} A)(250 \cdot 10^3 \Omega) = 1 - 0.25 = 0.75V. \qquad \square$$

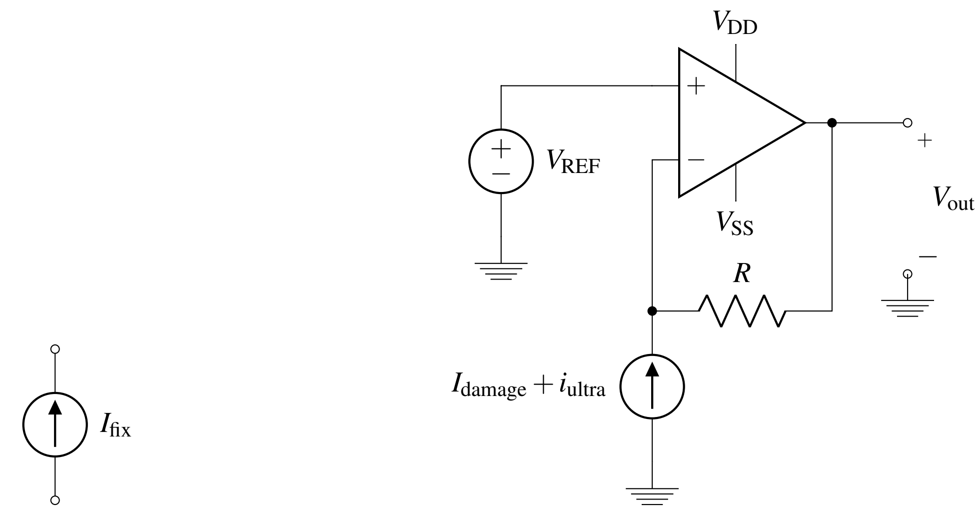
(c) (5 points) Unfortunately, after a few hours of successful ultrasound sensing, our sensor got damaged. It now constantly generates a huge background current, I_{damage}. So when an ultrasonic wave hits it, the sensor produces $I_{\text{damage}} + i_{\text{ultra}}$, as shown in Fig 3(b). When no ultrasonic wave hits it, the sensor produces just I_{damage}. However, the huge background current causes our output to constantly be $V_{\text{out}} = V_{SS}$, so we are not able to tell whether an ultrasonic wave is present or not.

We would like to fix this in our circuit by canceling the background current and retaining only the useful signal. For this purpose we are going to use a current source, I_{fix}, shown in Fig. 3(a), whose value we can choose. **How would you connect this current source in your circuit and what value would you pick for it?** Redraw the entire circuit with the new current source, I_{fix}, added and give the value of I_{fix} in terms of $I_{\text{damage}}, i_{\text{ultra}}, R, V_{\text{REF}}$. **Explain how your design works.**

Solutions:

We want to have only i_{ultra} flow through R. To achieve this we will insert the correcting current source in parallel with the input source $i_{\text{ultra}} + I_{\text{damage}}$ and set it at $I_{\text{fix}} = I_{\text{damage}}$ in opposite polarity, so that KCL gives:

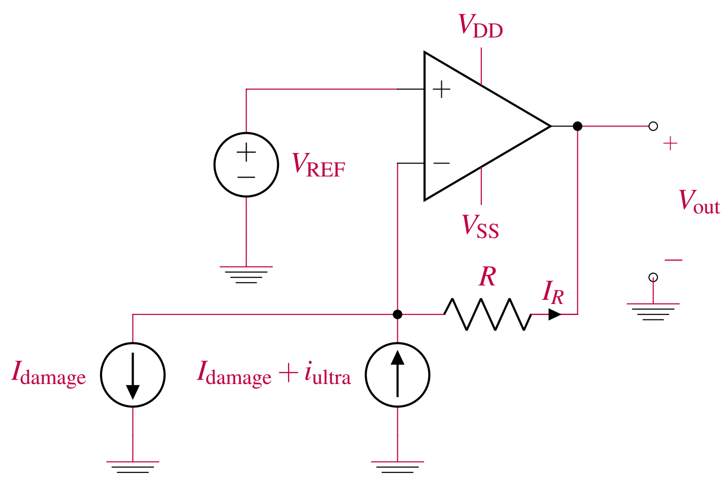
$$I_{\text{damage}} + i_{\text{ultra}} = I_{\text{damage}} + I_R \quad \longrightarrow \quad I_R = i_{\text{ultra}} \qquad \square$$



(a) Constant current source I_{fix}.

(b) Damaged transresistance amplifier circuit.

Figure 3: Circuits detailing the transresistance amplifier design, including the background signal I_{damage}.



3. Saving Lives with Op-Amps (19 points)

An electrocardiogram, or ECG, is a medical device used to detect electrical signals in your heart. Typically, the voltage signal from the human heart is only 1×10^{-3} V at maximum. However, in order for healthcare professionals to properly interpret ECGs, these signals need to be amplified so that abnormalities are more obvious. In this problem we will do so by using **ideal op-amps**.
Note: Assume that $V_{DD} = -V_{SS}$ in all subparts.

(a) (3 points) We need to amplify the voltage signal recorded by the electrodes V_{in} by a factor of 1000. Using the op-amp in Figure 4 below and 2 resistors, **draw a circuit that achieves $V_{\text{out}} = 1000 \cdot V_{\text{in}}$. Write an equation for V_{out} in terms of V_{in} and the resistor(s), label the resistors you use (i.e. R_1, R_2), and choose their values**. You should redraw the entire circuit in your answer sheet, but there is no need to draw the human as long as you label V_{in}. Clearly explain and show your work.

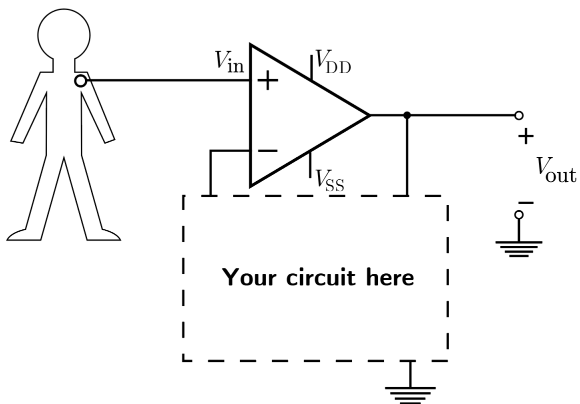


Figure 4: Unfinished ECG amplification circuit.

Solutions:
To achieve this input-output relationship we need to use a non-inverting amplifier like the one shown in Fig. 5, which was analyzed in lecture and gives: $V_{\text{out}} = V_{\text{in}}(1 + \frac{R_{\text{top}}}{R_{\text{bottom}}})$.
To get $V_{\text{out}} = 1000V_{\text{in}}$, we need to size R_{top} and R_{bottom} such that $R_{\text{top}} = 999R_{\text{bottom}}$.
One such option is selecting $R_{\text{top}} = 999 \cdot 10^3\Omega$ and $R_{\text{bottom}} = 1 \cdot 10^3\Omega$.

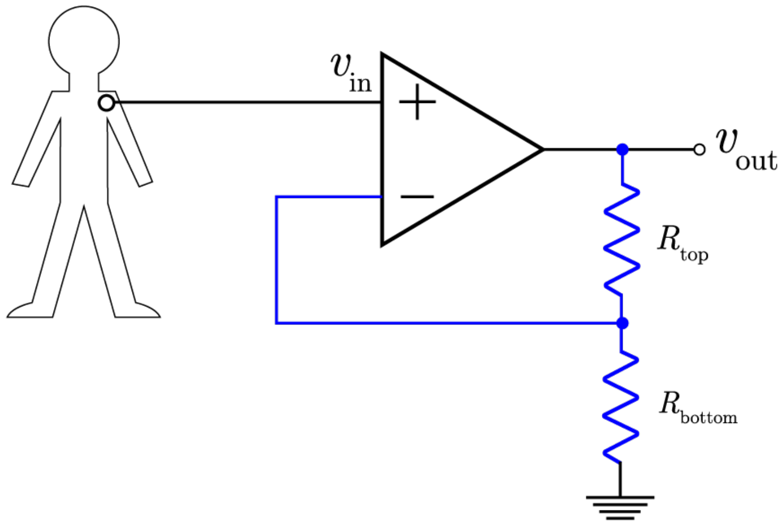


Figure 5: Complete ECG amplification circuit.

(b) (4 points) A friend of yours is also working on an ECG amplification circuit, and shows you their design in Figure 6. Their design uses $R_{\text{electrode}} = 1 \cdot 10^3\Omega$, $R_1 = 1 \cdot 10^3\Omega$, and $R_2 = 1 \cdot 10^6\Omega$. They claim their circuit gives, $V_{\text{out}} = -1000 \cdot V_{\text{in}}$. **Is their claim true?**

- **If yes, justify why.**
- **If no, how would you choose the value of R_2 to achieve $V_{\text{out}} = -1000 \cdot V_{\text{in}}$, assuming that both $R_{\text{electrode}}$ and R_1 are fixed at $R_{\text{electrode}} = R_1 = 1 \times 10^3\Omega$?** Clearly show your work, and justify your answers.

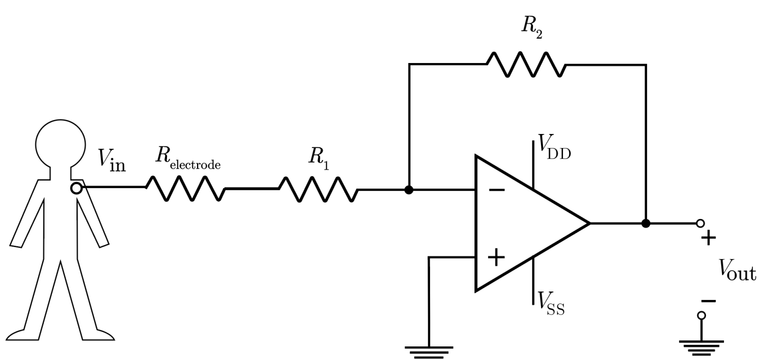


Figure 6: An alternative ECG op-amp circuit.

Solutions: This is the inverting amplifier topology analyzed in lecture. Using equivalence to lump R_1 and $R_{\text{electrode}}$ together we get that:

$$V_{\text{out}} = -\frac{R_2}{R_1 + R_{\text{electrode}}} V_{\text{in}} = - V_{\text{in}} = -500 V_{\text{in}} \neq -1000 V_{\text{in}}\,.$$

So their claim is not true.

Since we have

$$V_{\text{out}} = -\frac{R_2}{R_1 + R_{\text{electrode}}} V_{\text{in}}$$

and R_1, $R_{\text{electrode}}$ are fixed at $R_{\text{electrode}} = R_1 = 1 \times 10^3\Omega$, we need to set $R_2 = 2\text{M}\Omega = 2 \times 10^6\Omega$ in order to get $V_{\text{out}} = -1000 V_{\text{in}}$.

(c) (4 points) Another configuration often used by healthcare professionals is to attach one electrode to the heart (recording its electrical signal, V_{in}) and another electrode to the right leg to serve as a reference voltage, as shown in Figure 7. **What is the output voltage, V_{out}, as a function of V_{in}, V_{RL}, R_{bottom}, and R_{top}?** Clearly show your work.

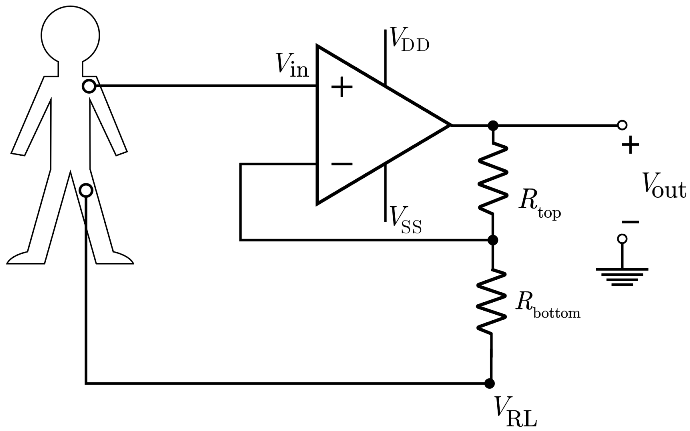


Figure 7: Alternative op-amp ECG topology.

Solutions:

Method 1: Superposition

We can find the output of this circuit by treating V_{RL} as a second input and apply superposition: Zeroing out V_{in} first and looking at V_{RL} we get the folllowing equivalent ckt:

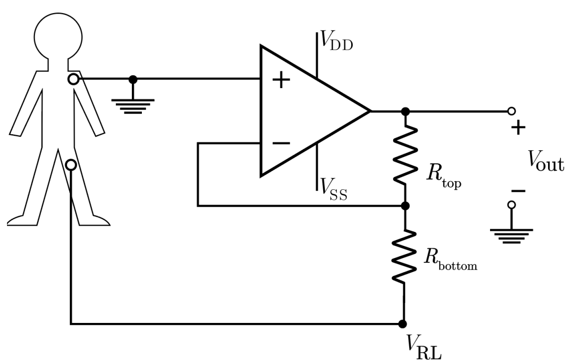


Figure 8: Alternative op-amp ECG topology with V_{in} zeroed-out

We can see that this is an inverting amplifier, so the output of the circuit is:

$$V_{\text{out},V_{RL}} = -\frac{R_{top}}{R_{bottom}} V_{\text{RL}} \, .$$

Next we zero-out and look at the output due to V_{in} only:

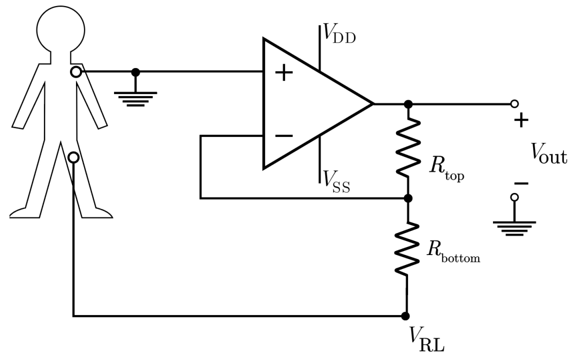


Figure 9: Alternative op-amp ECG topology with V_{RL} zeroed-out

Which is a non-inverting amplifier whose output is:

$$V_{\text{out},V_{\text{in}}} = (1 + \frac{R_{top}}{R_{bottom}})V_{\text{in}}\,.$$

Applying superposition, we get:

$$V_{out} = V_{\text{out},V_{\text{in}}} + V_{\text{out},V_{RL}} = V_{\text{in}}(1 + \frac{R_{top}}{R_{bottom}}) - V_{RL}(\frac{R_{top}}{R_{bottom}})\,.$$

Method 2: NVA

Alternatively, we can use NVA & golden rules to find the output of this circuit: Since this ckt is in negative feedback, golden rule #2 gives $u_- = V_{\text{in}}$, while KCL at u_- gives:

$$\begin{aligned} I_{R_{\text{bottom}}} &= I_{R\text{top}} \\ \Rightarrow \frac{V_{\text{RL}} - u_-}{R_{\text{bottom}}} &= \frac{u_- - V_{\text{out}}}{R_{\text{top}}} \\ \Rightarrow \frac{V_{\text{RL}} - V_{\text{in}}}{R_{\text{bottom}}} &= \frac{V_{\text{in}} - V_{\text{out}}}{R_{\text{top}}}\,. \end{aligned}$$

Solving for V_{out}, we get:

$$V_{out} = V_{\text{in}}(1 + \frac{R_{top}}{R_{bottom}}) - V_{RL}(\frac{R_{top}}{R_{bottom}})\,.$$

(d) (8 points) Even after amplification, certain peaks of your ECG signal are too low to be discerned. You want to sample them and amplify them a bit more. To this end, you use the circuit in Figure 10. The circuit cycles through two phases: in phase 1, switches labeled ϕ_1 are ON and ϕ_2 are OFF, while in phase 2, switches labeled ϕ_2 are ON and ϕ_1 are OFF. **Calculate the output voltage, V_{out}, during phase 2, after steady state has been reached, in terms of C_1, C_2 and V_{in}.** Clearly show your work.

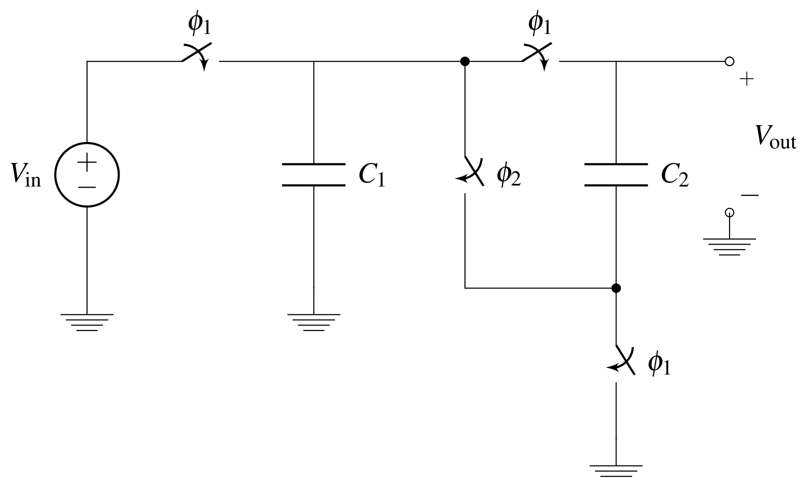
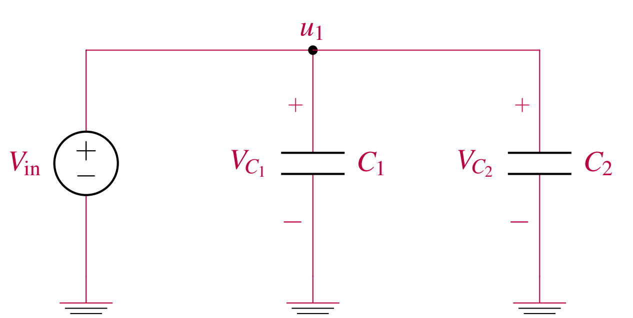


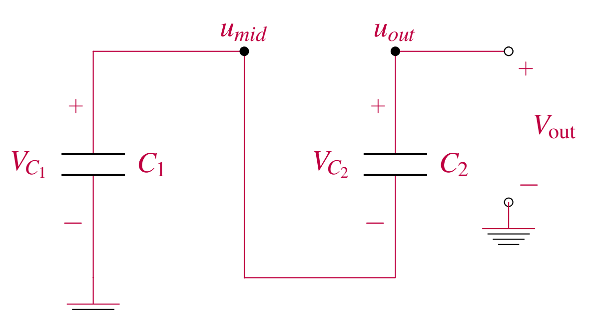
Figure 10: Switch capacitor voltage boosting circuit.

Solutions:

The equivalent circuit during phase 1 is:



The equivalent circuit during phase 2 is:



We can see that during phase 2 there are two floating nodes: u_{mid} and u_{out}.

Node u_{mid} is connected to the "+" plate of C_1 and the "−" plate of C_2. We will first calculate the charge on those plates in phase 1:

$$Q_{u_{mid}}^{\phi_1} = C_1 V_{\text{in}} - C_2 V_{\text{in}}.$$

And then the charge on that node during phase 2:

$$Q_{u_{mid}}^{\phi_2} = C_1 u_{mid} - C_2(u_{out} - u_{mid}) \,.$$

Equating the two we get:

$$C_1 V_{\text{in}} - C_2 V_{\text{in}} = C_1 u_{mid} - C_2(u_{out} - u_{mid}) \,. \tag{1}$$

Next, we will look at node u_{out}, which is connected to the "+" plate of C_2 during phase 2. For the charge stored on that u_{out} during phase 1 we have:

$$Q_{u_{out}}^{\phi_1} = C_2 V_{\text{in}} \,.$$

The charge on that node during phase 2:

$$Q_{u_{out}}^{\phi_2} = C_2(u_{out} - u_{mid}) \,.$$

Equating the two we get:

$$C_2 V_{\text{in}} = C_2(u_{out} - u_{mid}) \Rightarrow V_{\text{in}} = u_{out} - u_{mid} \,. \tag{2}$$

Plugging (2) into (1) we get:

$$\begin{aligned} C_1 V_{\text{in}} - C_2 V_{\text{in}} &= C_1 u_{mid} - C_2 V_{\text{in}} \\ \Rightarrow V_{\text{in}} &= u_{mid} \,. \end{aligned} \tag{3}$$

Finally, plugging (3) into (2) we get:

$$V_{\text{out}} = u_{out} - 0 = 2V_{\text{in}} \,.$$

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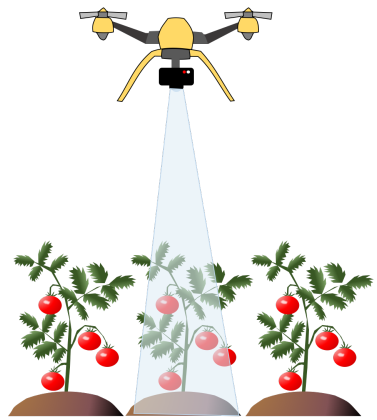
4. Hyperspectral Classification of Tomatoes (14 points)

You're a high-tech farmer who just bought a new hyperspectral sensor to monitor your crops.
NOTE: You do not need to understand how a hyperspectral sensor works to solve this problem.

You attach the sensor to a drone and fly it over your crops, taking measurements of the hyperspectral signature for different points along the field. You want to use these measurements to identify which crops are healthy and which crops are getting sick. Your sensor gives you a spectral signature for each plant as a length 5 vector, where each entry of the vector represents a different frequency. Scientists have determined that healthy versus sick tomato plants will have the following spectral signatures as shown in Figure 11.



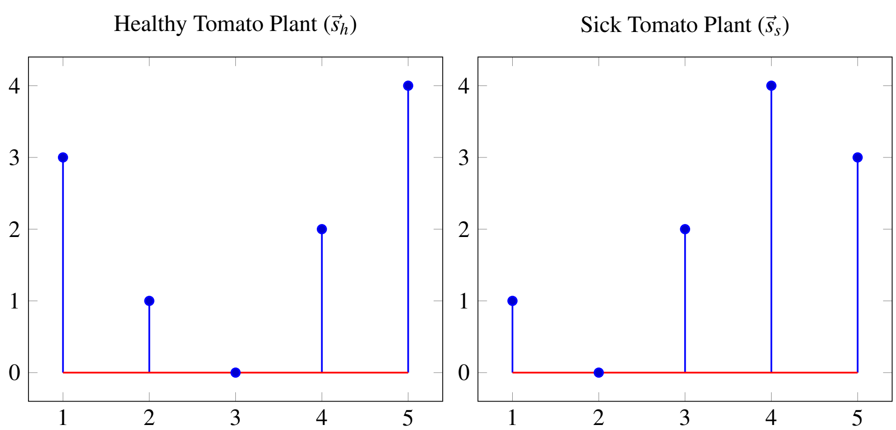


Figure 11: Spectral signature for healthy tomato plant ($\vec{s}_h$) and sick tomato plant ($\vec{s}_s$).

They can also be represented in vector form:

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

(a) (6 points) Using your spectral sensor, you measure the following spectral signature for one of your tomato plants as shown in Figure 12. This measurement has some noise in it.

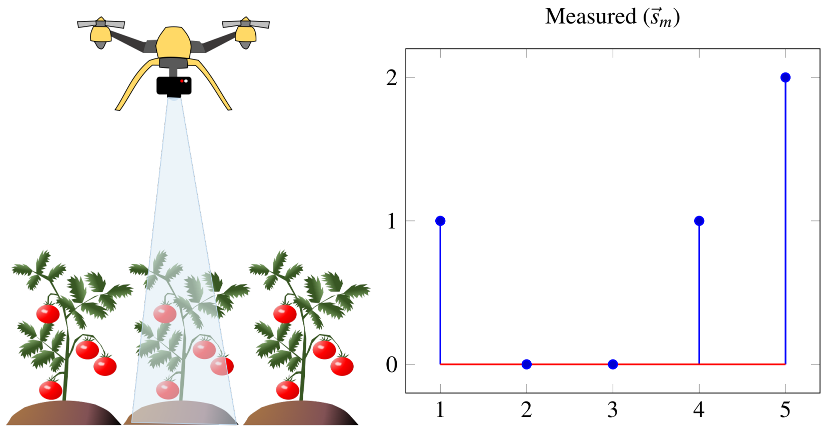


Figure 12: Spectral signature for the measurement, $\vec{s}_m$.

The spectral signatures for healthy, sick, and your measured tomato plants can also be represented in vector form as

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \qquad \vec{s}_m = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Since spectral signatures never exactly match, the standard procedure is to calculate the angle between signature vectors to determine how close they are. **Compute the angle between $\vec{s}_m$ and $\vec{s}_h$ and the angle between $\vec{s}_m$ and $\vec{s}_s$. Is your measured vector closer to the sick plants or the healthy plants?** Classify your plant's health based on the angle between your measured spectral signature ($\vec{s}_m$) and the known spectral signatures, ($\vec{s}_h$, $\vec{s}_s$). Show your work and justify your answer.
NOTE: Table 2 can be helpful for finding the angles.

Table 2: Cosine Table

$\cos(\theta)$	$\theta(^\circ)$
$\frac{9}{\sqrt{180}}$	47.87
$\frac{10}{\sqrt{180}}$	41.81
$\frac{11}{\sqrt{180}}$	34.93
$\frac{12}{\sqrt{180}}$	26.57
$\frac{13}{\sqrt{180}}$	14.31

Solutions:
The solution to this problem is to use the inner product formula to compute the angles:

$$cos(\theta) = \frac{< \vec{s}_1, \vec{s}_2 >}{||\vec{s}_1|| ||\vec{s}_2||}.$$

Comparing the measurement with the spectral signature for the healthy plant, we get:

$$\begin{aligned} cos(\theta_1) &= \frac{< \vec{s}_m, \vec{s}_h >}{||\vec{s}_m|| ||\vec{s}_h||} \\ &= \frac{< \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} >}{|| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} |||| \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} ||} \\ &= \frac{13}{\sqrt{6} * \sqrt{30}} \\ &= \frac{13}{\sqrt{180}}. \end{aligned}$$

Comparing the measurement with the spectral signature for the sick plant, we get:

$$\begin{aligned} cos(\theta_2) &= \frac{< \vec{s}_m, \vec{s}_s >}{||\vec{s}_m|| ||\vec{s}_s||} \\ &= \frac{< \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix} >}{|| \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} |||| \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix} ||} \\ &= \frac{11}{\sqrt{6} * \sqrt{30}} \\ &= \frac{11}{\sqrt{180}}. \end{aligned}$$

From the cosine table, we get that $\theta_1 = 14.31°$ and $\theta_2 = 34.93°$. The measured tomato plant has a smaller angle with the healthy plants. Therefore, the plant is healthy.

(b) (4 points) It's a windy day and the drone got pushed as it was taking a measurement, so now the measurement has a linear combination of measurements for several different tomato plants (some of which are healthy and some of which are sick). So your measurement is

$$\vec{s}_m = \alpha \vec{s}_h + \beta \vec{s}_s + \vec{e}, \tag{4}$$

where $\vec{e}$ represents an error vector that is unknown.

The values you get for your measurement are:

$$\vec{s}_m = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 10 \\ 10 \end{bmatrix}.$$

The measurement is also shown in Figure 13.

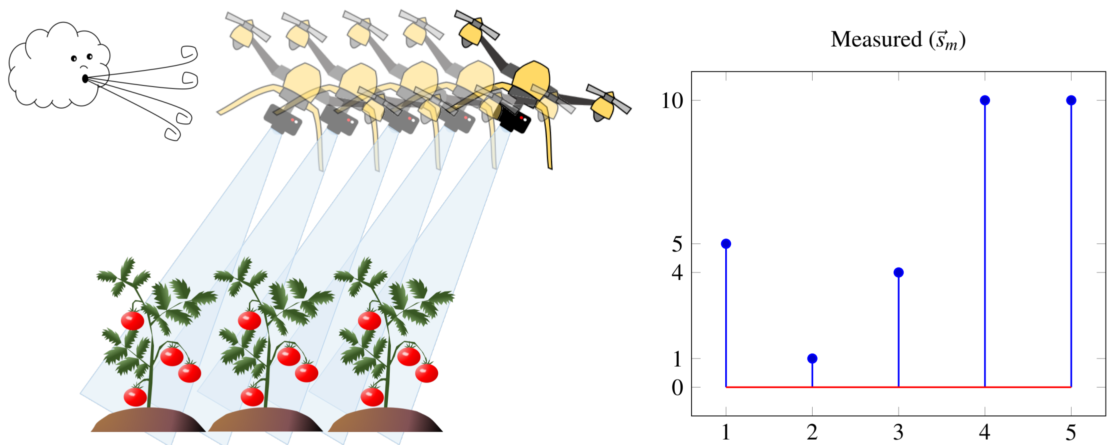


Figure 13: Spectral signature for your measurement, $\vec{s}_m$.

Recall that

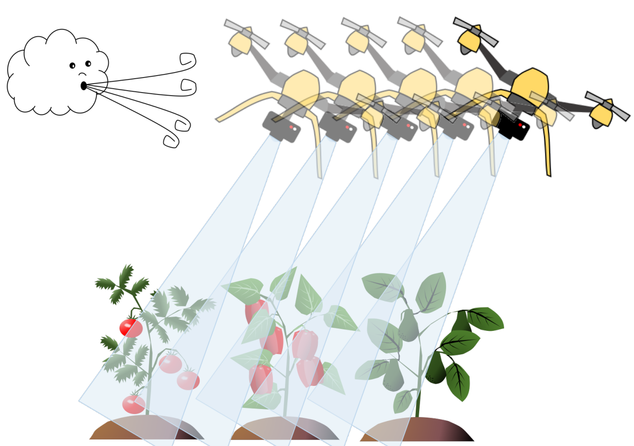
$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

You want to identify the unknowns α and β. **Write a least squares problem in the format $\mathbf{A}\vec{x} = \vec{b}$ to identify the unknowns α and β. Show your work.** You do not have to solve for α and β.

Solutions:

$$\vec{s}_m = \mathbf{A}\vec{x}$$
$$= \begin{bmatrix} | & | \\ \vec{s}_h & \vec{s}_s \\ | & | \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$
$$\begin{bmatrix} 5 \\ 1 \\ 4 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & 2 \\ 2 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

(c) (4 points) Your drone got pushed by the wind again, but this time it was while it was taking a measurement on the border of three adjacent fields - your tomato, pepper, and avocado fields.



Tomato, pepper, and avocado plants have unique spectral signatures with a length of 5. The notations are described as the following:

- $\vec{s}_h$ and $\vec{s}_s$ represent the spectral signatures of healthy and sick tomato plants

- $\vec{s}_{ph}$ and $\vec{s}_{ps}$ represent the spectral signatures of healthy and sick pepper plants
- $\vec{s}_{ah}$ and $\vec{s}_{as}$ represent the spectral signatures of healthy and sick avocado plants

Your measurement is now a linear combination of 6 possible spectral signatures:

$$\vec{s}_m = \alpha_1 \vec{s}_h + \beta_1 \vec{s}_s + \alpha_2 \vec{s}_{ph} + \beta_2 \vec{s}_{ps} + \alpha_3 \vec{s}_{ah} + \beta_3 \vec{s}_{as}. \tag{5}$$

Here $\alpha_1, \alpha_2, \alpha_3$ are the unknown weights of healthy tomato, pepper, and avocado plants respectively. $\beta_1, \beta_2, \beta_3$ are the unknown weights of sick tomato, pepper, and avocado plants respectively. **Is it possible to uniquely determine the weights of healthy/sick tomatoes, peppers, and avocados from your measurement in equation 5? Why or why not? Show your work and justify your answer.**

Solutions:

We write out equation (5) into a matrix-vector form:

$$\vec{s}_m = \mathbf{A}\vec{x}$$

$$\Rightarrow \vec{s}_m = \begin{bmatrix} | & | & | & | & | & | \\ \vec{s}_h & \vec{s}_{ph} & \vec{s}_{ah} & \vec{s}_s & \vec{s}_{ps} & \vec{s}_{as} \\ | & | & | & | & | & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}.$$

We will not be able to uniquely determine the unknown weights. This problem is underdetermined and has many possible solutions, since the matrix is of size 5×6 and we therefore have more unknowns than we have equations.

5. Cross-correlation (28 points)

We are building our own Acoustic Positioning System.
NOTE: The signatures $\vec{s}_1$, $\vec{s}_2$ in each sub-part are different; each prompt is independent from the others.

(a) (6 points) We have two signatures/gold codes of length-5, given by $\vec{s}_1$ and $\vec{s}_2$ as in Figure 14. So far we have numerically computed their linear cross-correlation $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$, yet a few entries have been tragically lost! Fortunately we can compute these omitted terms by hand. **Please compute the missing cross-correlation values at shifts $k=-1$ and $k=+2$. Show your work and justify your answer.**

$$\vec{s}_1 = \begin{bmatrix} +1 \\ 0 \\ -1 \\ 0 \\ +1 \end{bmatrix} \qquad \vec{s}_2 = \begin{bmatrix} +1 \\ +1 \\ 0 \\ -1 \\ +1 \end{bmatrix}.$$

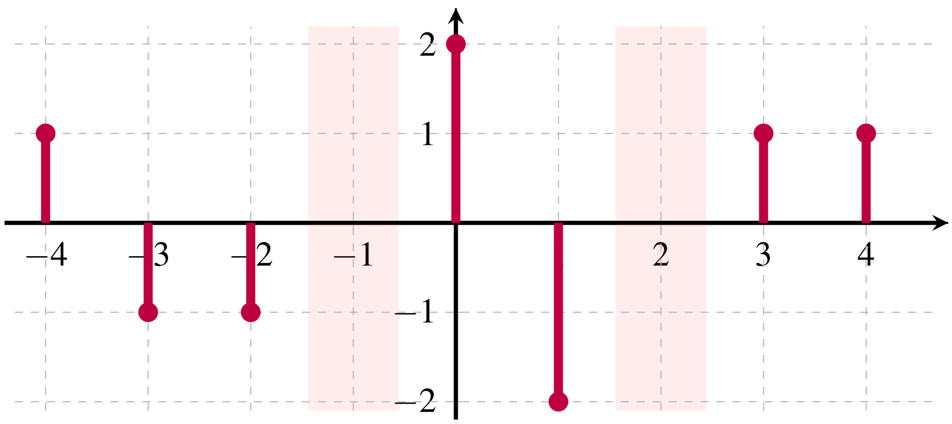


Figure 14: Linear cross-correlation plot of the two signals $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$. The x-axis represents the shift.

Solutions:
To find the cross-correlation value at shift k=-1, we may directly compute it from $\text{Corr}_{\vec{s}_1}(\vec{s}_2)[-1] = \sum_n \vec{s}_1[n]\ \vec{s}_2[n+1]$ over all nonzero terms, so for $n = 1,2,3,4$:

$$\begin{aligned} \sum_{n=1}^{4} \vec{s}_1[n]\ \vec{s}_2[n+1] &= 1\cdot 1 + 0\cdot 0 + -1\cdot -1 + 0\cdot 1 \\ &= 2\,. \end{aligned}$$

Next we compute $\text{Corr}_{\vec{s}_1}(\vec{s}_2)[+2] = \sum_n \vec{s}_1[n]\ \vec{s}_2[n-2]$ but now over just $n = 3,4,5$:

$$\begin{aligned} \sum_{n=1}^{3} \vec{s}_1[n]\ \vec{s}_2[n-2] &= -1\cdot 1 + 0\cdot 1 + 1\cdot 0 \\ &= -1\,. \end{aligned} \qquad \square$$

(b) (4 points) We are trying out some new codes $\vec{s}_1$ and $\vec{s}_2$. We only know that the codes are normalized ($\langle \vec{s}_1, \vec{s}_1 \rangle = 1$, $\langle \vec{s}_2, \vec{s}_2 \rangle = 1$) and their inner-product is $\langle \vec{s}_1, \vec{s}_2 \rangle = 0.3$. During our test we have received the signal $\vec{r} = \frac{1}{2}\vec{s}_1 + \frac{1}{3}\vec{s}_2$. Without knowing any more information about our codes, **compute $\text{Corr}_{\vec{r}}(\vec{s}_1)$ at the shift $k = 0$. Show your work and justify your answer.**

Solutions:

This is possible only because at $k = 0$, the correlation reduces to an inner-product: $\text{Corr}_{\vec{r}}(\vec{s}_1)[k=0] = \langle \vec{r}, \vec{s}_1 \rangle$. Since the inner-product is a linear operation, we can expand out $\vec{r}$ and find the result.

$$
\begin{aligned}
\text{Corr}_{\vec{r}}(\vec{s}_1)[k=0] &= \langle \vec{r}, \vec{s}_1 \rangle \\
&= \langle \ \tfrac{1}{2}\vec{s}_1 + \tfrac{1}{3}\vec{s}_2 \ , \ \vec{s}_1 \ \rangle \\
&= \tfrac{1}{2} \langle \vec{s}_1, \vec{s}_1 \rangle \ + \ \tfrac{1}{3} \langle \vec{s}_1, \vec{s}_2 \rangle \\
&= \tfrac{1}{2}\, 1.0 \ + \ \tfrac{1}{3}\, 0.3 \\
&= 0.6 \qquad \square
\end{aligned}
$$

(c) (4 points) We again have two new signals $\vec{s}_1$ and $\vec{s}_2$, and are now given the plot of $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$ as shown in Figure 15. Our receiver identified a signal $\vec{r}$ which we know to be related to the code $\vec{s}_2$ by some scaling, shifting, and/or reflection. However, we only know the linear cross-correlation $\text{Corr}_{\vec{s}_1}(\vec{r})$ as shown in Figure 16. **Can you express $\vec{r}$ in terms of $\vec{s}_2$? Show your work and justify your answer.**

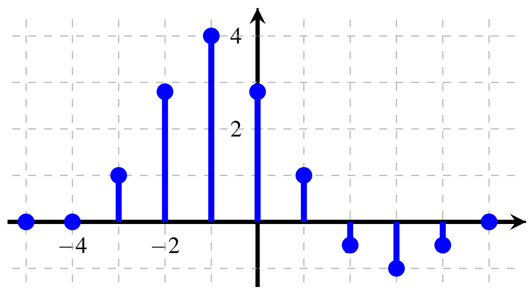


Figure 15: Linear cross-correlation plots for $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$.

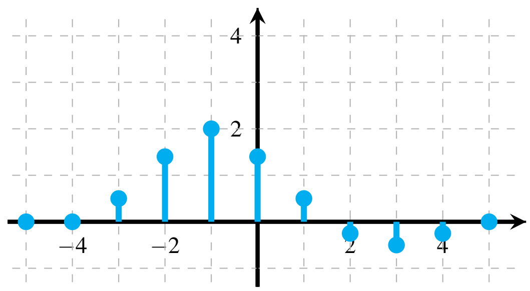


Figure 16: Linear cross-correlation plots for $\text{Corr}_{\vec{s}_1}(\vec{r})$.

Solutions:
By inspection of Figure 16, the correlation plot of $\vec{s}_1$ with $\vec{r}$ is the same as the correlation plot of $\vec{s}_1$ with $\vec{s}_2$, only altered by a vertical scaling of $1/2$. This is most evident at the $k = -1$ shift, where $\text{Corr}_{\vec{s}_1}(\vec{r})[-1] = 2$ and $\text{Corr}_{\vec{s}_1}(\vec{s}_2)[-1] = 4$.

So far we've concluded that $\text{Corr}_{\vec{s}_1}(\vec{r}) = \frac{1}{2}\text{Corr}_{\vec{s}_1}(\vec{s}_2)$.
Next we must recognize that correlation is a linear operation, so we can see

$$\text{Corr}_{\vec{s}_1}\left(\frac{1}{2}\vec{s}_2\right)[k] = \sum_n \vec{s}_1[n]\,\frac{1}{2}\vec{s}_2[n-k] = \frac{1}{2}\text{Corr}_{\vec{s}_1}(\vec{s}_2)[k].$$

Thus the relationship of $\vec{r}$ with respect to $\vec{s}_2$ is

$$\vec{r}[n] = \frac{1}{2}\,\vec{s}_2[n]. \qquad \square$$

(d) (4 points) With a little effort we managed to create two *good* gold codes of length 100, $\vec{s}_1$ and $\vec{s}_2$. The linear cross-correlation of $\vec{s}_1$ and $\vec{s}_2$ is small at all shifts while the autocorrelation of each signal is also small, except at shift $k = 0$. We receive our first signal $\vec{r}$ which we know to be a combination of both codes

$$\vec{r}[n] \;=\; \vec{s}_1[n-k_1] \;+\; \vec{s}_2[n-k_2]. \tag{6}$$

The linear cross-correlation $\text{Corr}_{\vec{r}}(\vec{s}_1)$ has been computed and plotted in Figure 17, and similarly $\text{Corr}_{\vec{r}}(\vec{s}_2)$ is plotted in Figure 18. **Determine the shifts for $\vec{s}_1$ and $\vec{s}_2$ in the received signal $\vec{r}$, i.e. solve for k_1 and k_2 in equation (6). Explain your answer.**
Note: Don't worry too much about identifying the exact value for k_1 and k_2. As long as your answer is reasonable, you will receive full credit.

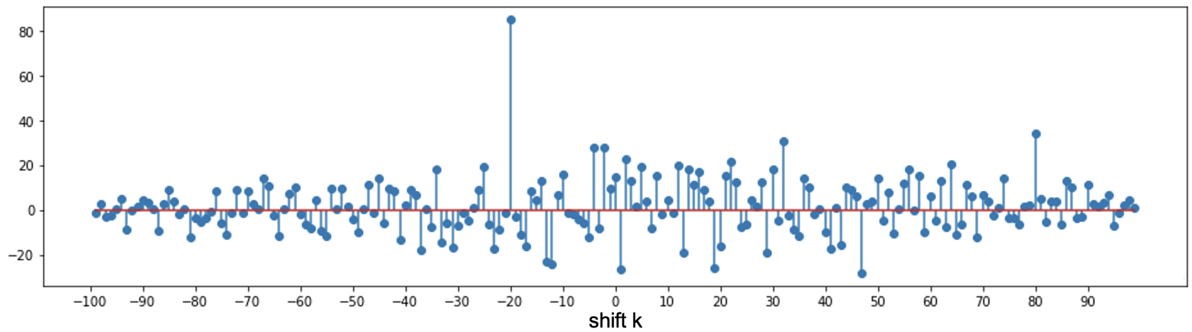


Figure 17: Linear cross-correlation plots for $\text{Corr}_{\vec{r}}(\vec{s}_1)$.

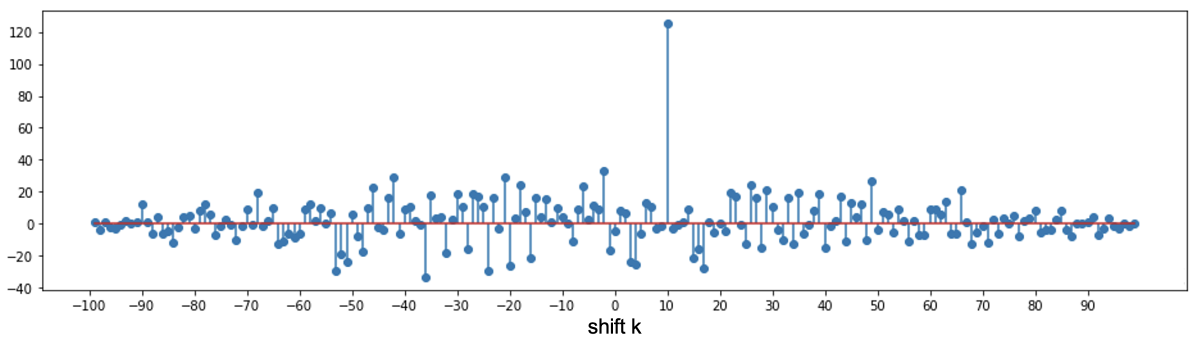


Figure 18: Linear cross-correlation plots for $\text{Corr}_{\vec{r}}(\vec{s}_2)$.

Solutions: Let us start substituting $\vec{r}[n] = \vec{s}_1[n-k_1] + \vec{s}_2[n-k_2]$ into our correlation definitions.

$$\text{Corr}_{\vec{r}}(\vec{s}_1)[k] = \sum_n \vec{r}[n]\,\vec{s}_1[n-k] = \sum_n \vec{s}_1[n-k_1]\,\vec{s}_1[n-k] + \cancel{\sum_n \vec{s}_2[n-k_2]\,\vec{s}_1[n-k]}$$
$$\text{Corr}_{\vec{r}}(\vec{s}_2)[k] = \sum_n \vec{r}[n]\,\vec{s}_2[n-k] = \cancel{\sum_n \vec{s}_1[n-k_1]\,\vec{s}_2[n-k]} + \sum_n \vec{s}_2[n-k_2]\,\vec{s}_2[n-k]$$

We approximate the cancellation of terms since codes $\vec{s}_1$ and $\vec{s}_2$ have a small cross-correlation.

From Figure 17 we note for $\text{Corr}_{\vec{r}}(\vec{s}_1)$ the only significant peak occurs at shift $k = -20$. Since the auto-correlation $\text{Corr}_{\vec{s}_1}(\vec{s}_1)$ is peaked at the zero-shift, it must be (based on the top equation) that $k_1 = k = -20$. With similar reasoning (in regards to Figure 18) we identify $k_2 = +10$.

Thus we arrive at our solution for the received signal:

$$\vec{r}[n] = \vec{s}_1[n+20] + \vec{s}_2[n-10].$$

(e) (4 points) It appears that making codes orthogonal to each other improves the robustness of our Acoustic Positioning System. Knowing this, we want to use our knowledge of projections to write our first code as $\vec{s}_1 = \vec{a} + \vec{b}$, where $\langle \vec{b}, \vec{s}_2 \rangle = 0$ and $\vec{a} = \alpha\, \vec{s}_2$ (for some constant α) as illustrated in Figure 19. **Compute α and $\vec{b}$ in terms of $\vec{s}_1$ and $\vec{s}_2$. Show your work and justify your answer.**

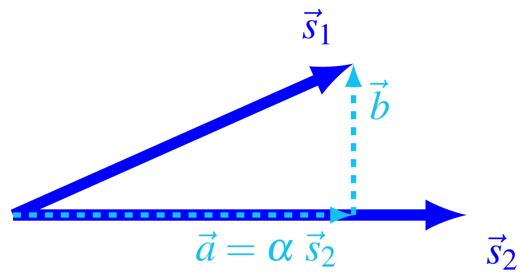


Figure 19: 2D figure of $\vec{s}_1 = \vec{a} + \vec{b}$.

Solutions:
From the setup we may observe $\vec{a}$ is the projection of $\vec{s}_1$ onto $\vec{s}_2$.

$$\vec{a} \;=\; \text{proj}_{\vec{s}_2}(\vec{s}_1) \;=\; \left(\frac{\langle \vec{s}_1, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}\right) \vec{s}_2 \qquad \longrightarrow \qquad \alpha \;=\; \frac{\langle \vec{s}_1, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}$$

Acquiring the new orthogonal code $\vec{b}$ follows from our result above

$$\vec{b} \;=\; \vec{s}_1 - \vec{a} \;=\; \vec{s}_1 - \left(\frac{\langle \vec{s}_1, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}\right) \vec{s}_2. \qquad \square$$

– Alternate Method –
The constant α can also be determined algebraically using the fact that $\vec{b}$ is orthogonal to $\vec{s}_2$

$$\langle \vec{s}_1, \vec{s}_2 \rangle \;=\; \alpha \langle \vec{s}_2, \vec{s}_2 \rangle \;+\; \cancel{\langle \vec{b}, \vec{s}_2 \rangle}.$$

Thus we can write α in terms of $\vec{s}_1$ and $\vec{s}_2$: $\alpha \;=\; \frac{\langle \vec{s}_1, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}$.
Finding $\vec{b}$ from this point follows identically as shown above

$$\vec{b} \;=\; \vec{s}_1 - \vec{a} \;=\; \vec{s}_1 - \alpha\, \vec{s}_2 \;=\; \vec{s}_1 - \left(\frac{\langle \vec{s}_1, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}\right) \vec{s}_2. \qquad \square$$

(f) (6 points) After optimizing two orthogonal codes $\vec{s}_1$ and $\vec{s}_2$ (i.e. $\langle \vec{s}_1, \vec{s}_2 \rangle = 0$), we would next like to include another code $\vec{s}_3$ and make it orthogonal to $\vec{s}_1$ and $\vec{s}_2$. We can start by writing $\vec{s}_3$ as $\vec{s}_3 = \vec{a} + \vec{b}$, such that $\vec{a}$ belongs to the span$\{\vec{s}_1, \vec{s}_2\}$ and $\vec{b}$ is orthogonal to span$\{\vec{s}_1, \vec{s}_2\}$, i.e. $\langle \vec{b}, \vec{s}_1 \rangle = 0$

and $\langle \vec{b}, \vec{s}_2 \rangle = 0$. **Use the idea of projections to write both $\vec{a}$ and $\vec{b}$ in terms of $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$, and inner-products thereof. (For full credit your final answer may not contain matrices nor matrix-vector products).** Show your work and justify your answer.

Solutions:

In least-squares method, the minimizing solution $\hat{\vec{x}}$ to the system $\mathbf{A}\vec{x} = \vec{y}$ will result in $\mathbf{A}\hat{\vec{x}}$ producing the projection of vector $\vec{y}$ onto the column space of $\mathbf{A}$, in which the error vector $\vec{e} = \vec{y} - \mathbf{A}\hat{\vec{x}}$ is orthogonal to the span of the column vectors in $\mathbf{A}$. In this problem's context, we need to acquire $\vec{b} \equiv \vec{s}_3 - \mathbf{A}\hat{\vec{x}}$, in which the original signal $\vec{s}_3$ must be projected onto $\mathbf{A} \equiv \begin{bmatrix} \uparrow & \uparrow \\ \vec{s}_1 & \vec{s}_2 \\ \downarrow & \downarrow \end{bmatrix}$. So we may apply our least-squares formula to acquire $\vec{b} = \mathbf{A}\hat{\vec{x}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{s}_3$. But since $\mathbf{A}$ has orthogonal columns, we can substantially simplify this expression:

$$
\begin{aligned}
\mathbf{A}\hat{\vec{x}} &= \begin{bmatrix} \uparrow & \uparrow \\ \vec{s}_1 & \vec{s}_2 \\ \downarrow & \downarrow \end{bmatrix} \left(\begin{bmatrix} \leftarrow & \vec{s}_1^{\,T} & \rightarrow \\ \leftarrow & \vec{s}_2^{\,T} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow \\ \vec{s}_1 & \vec{s}_2 \\ \downarrow & \downarrow \end{bmatrix} \right)^{-1} \begin{bmatrix} \leftarrow & \vec{s}_1^{\,T} & \rightarrow \\ \leftarrow & \vec{s}_2^{\,T} & \rightarrow \end{bmatrix} \vec{s}_3 \\
&= \begin{bmatrix} \uparrow & \uparrow \\ \vec{s}_1 & \vec{s}_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} ||\vec{s}_1||^2 & 0 \\ 0 & ||\vec{s}_2||^2 \end{bmatrix}^{-1} \begin{bmatrix} \leftarrow & \vec{s}_1^{\,T} & \rightarrow \\ \leftarrow & \vec{s}_2^{\,T} & \rightarrow \end{bmatrix} \vec{s}_3 \\
&= \begin{bmatrix} \uparrow & \uparrow \\ \vec{s}_1 & \vec{s}_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} ||\vec{s}_1||^2 & 0 \\ 0 & ||\vec{s}_2||^2 \end{bmatrix}^{-1} \begin{bmatrix} \langle \vec{s}_1, \vec{s}_3 \rangle \\ \langle \vec{s}_2, \vec{s}_3 \rangle \end{bmatrix} \\
&= \begin{bmatrix} \uparrow & \uparrow \\ \vec{s}_1 & \vec{s}_2 \\ \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} \frac{\langle \vec{s}_1, \vec{s}_3 \rangle}{\langle \vec{s}_1, \vec{s}_1 \rangle} \\ \frac{\langle \vec{s}_2, \vec{s}_3 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle} \end{bmatrix} \\
&= \left(\frac{\langle \vec{s}_1, \vec{s}_3 \rangle}{\langle \vec{s}_1, \vec{s}_1 \rangle} \right) \vec{s}_1 + \left(\frac{\langle \vec{s}_2, \vec{s}_3 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle} \right) \vec{s}_2 \,.
\end{aligned}
$$

Finally, we can subtract this projection from the original signal $\vec{s}_3$ to obtain an orthogonal code to both $\vec{s}_1$ and $\vec{s}_2$:

$$\vec{b} = \vec{s}_3 - \left(\frac{\langle \vec{s}_3, \vec{s}_1 \rangle}{\langle \vec{s}_1, \vec{s}_1 \rangle} \right) \vec{s}_1 - \left(\frac{\langle \vec{s}_3, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle} \right) \vec{s}_2. \qquad \square$$

– Alternate Method –

Since codes $\vec{s}_1$ and $\vec{s}_2$ are orthogonal (so the columns of $\mathbf{A}$ are orthogonal), we can simplify the projection onto $\mathbf{A}$ by performing these operations independently using the formula $\text{proj}_{\mathbf{A}}(\vec{s}_3) = \text{proj}_{\vec{s}_1}(\vec{s}_3) + \text{proj}_{\vec{s}_2}(\vec{s}_3)$. This was derived in discussion and *must be referenced explicitly to receive credit.* The solution becomes:

$$\begin{aligned}\vec{b} &= \vec{s}_3 - \text{proj}_{\vec{s}_1}(\vec{s}_3) - \text{proj}_{\vec{s}_2}(\vec{s}_3) \\ &= \vec{s}_3 - \left(\frac{\langle \vec{s}_3, \vec{s}_1 \rangle}{\langle \vec{s}_1, \vec{s}_1 \rangle}\right)\vec{s}_1 - \left(\frac{\langle \vec{s}_3, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}\right)\vec{s}_2. \qquad \square\end{aligned}$$

To verify this works, we directly compute the inner-product of $\vec{b}$ with the two original signals:

$$\begin{aligned}\langle \vec{b}, \vec{s}_1 \rangle &= \langle \vec{s}_3, \vec{s}_1 \rangle - \left(\frac{\langle \vec{s}_3, \vec{s}_1 \rangle}{\langle \vec{s}_1, \vec{s}_1 \rangle}\right)\langle \vec{s}_1, \vec{s}_1 \rangle - \left(\frac{\langle \vec{s}_3, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}\right)\cancel{\langle \vec{s}_2, \vec{s}_1 \rangle} \\ &= \langle \vec{s}_3, \vec{s}_1 \rangle - \langle \vec{s}_3, \vec{s}_1 \rangle = 0\end{aligned}$$

$$\begin{aligned}\langle \vec{b}, \vec{s}_2 \rangle &= \langle \vec{s}_3, \vec{s}_2 \rangle - \left(\frac{\langle \vec{s}_3, \vec{s}_1 \rangle}{\langle \vec{s}_1, \vec{s}_1 \rangle}\right)\cancel{\langle \vec{s}_1, \vec{s}_2 \rangle} - \left(\frac{\langle \vec{s}_3, \vec{s}_2 \rangle}{\langle \vec{s}_2, \vec{s}_2 \rangle}\right)\langle \vec{s}_2, \vec{s}_2 \rangle \\ &= \langle \vec{s}_3, \vec{s}_2 \rangle - \langle \vec{s}_3, \vec{s}_2 \rangle = 0.\end{aligned}$$

Thus $\vec{b}$ is orthogonal to both original codes.
(Notice this only worked because $\vec{s}_1$ and $\vec{s}_2$ are orthogonal!)

6. Warm for the Holidays (14 points)

Winter is coming, and both you and your roommate are in desperate need of electric heating eye pads to avoid overly dry eyes this holiday. Tragically the circuit for your eye pads broke, yet fortunately you've taken EECS16A and have come up with a clever fix by designing a voltage divider and a comparator circuit!

(a) (4 points) First you build a circuit that converts temperature change to voltage change. Your design is shown in Fig. 20. In your design you use two temperature dependent resistors, whose values are given by $R_0 + \alpha T$, $R_0 - \alpha T$, where R_0 is the resistor value at 0 degrees centigrade, α is a thermal coefficient, and T is the temperature of your eye pads.
What is the temperature dependent output voltage, V_T, of this circuit, as a function of V_s, R_0, α, and T? Is V_T a linear function of T? Clearly show all your work.

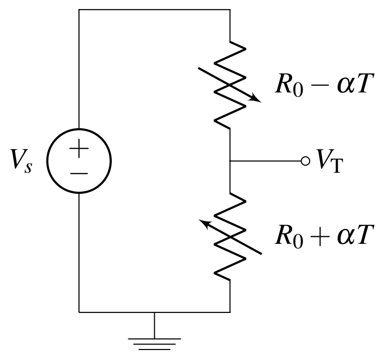


Figure 20: Temperature Sensing Circuit

Solutions: This circuit is essentially a voltage divider:

$$V_\text{T} = \frac{R_0 + \alpha T}{R_0 + \alpha T + R_0 - \alpha T} V_s = \frac{R_0 + \alpha T}{2R_0} V_s \,.$$

This is an affine function of T since there is an offset term in our final expression.

(b) (4 points) We want to use a comparator to turn the heat ON and OFF, and you set up the circuit in Fig. 21. You process the V_T to make $V_\text{in} = (1 - \frac{T}{T_0})$[Volts], where $T_0 = 30°\text{C}$. The heat will turn on when $V_\text{out} = V_\text{DD}$. **For what range of temperatures, T, is $V_\text{out} = V_\text{DD}$? Give your answer in terms of $°$C.** Clearly show all your work.

Solutions:

The output is equal to V_DD, when the voltage at the "+" terminal is larger than the "−" terminal of the comparator. The condition for that is:

$$V_\text{in} > 0 \rightarrow 1 - \frac{T}{T_0} > 0 \rightarrow T < T_0 \,,$$

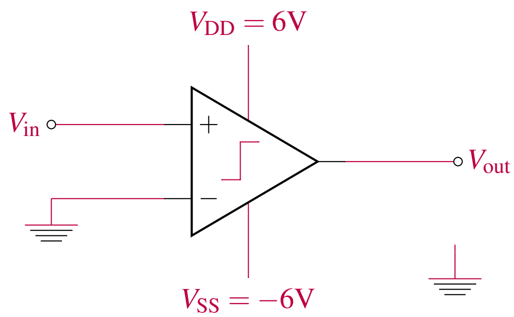


Figure 21: First attempt eye-pad control circuit.

which means that $V_{out} = V_{DD}$ (i.e. the heater turns ON) for all temperatures that are below 30°C.

(c) (6 points) Your TA, Moses, points out that just using the circuit in Figure 21 will cause your heat to turn ON and OFF due to very small fluctuations. Instead, he suggests analyzing the following circuit in Figure 22. **Find the voltage u_+ at the positive terminal of the comparator, as a function of V_{out}, R_1, R_2, and V_{ref}.** Clearly show all your work.

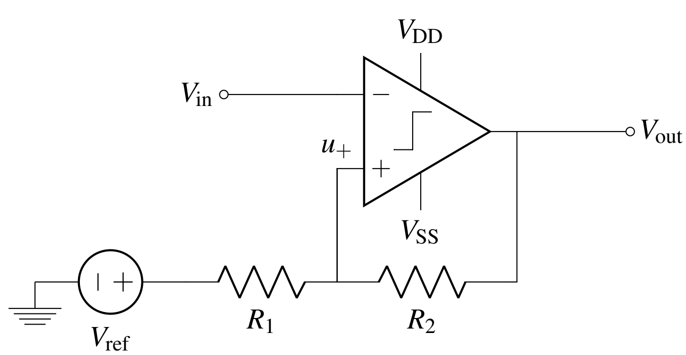
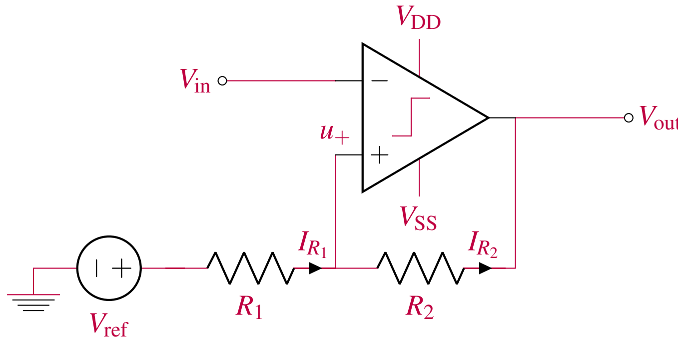


Figure 22: Proposed eye-pad control circuit.

Solutions:



A common misconception here was to assume that the circuit is in negative feedback. It is not, the output connects back to the '+' terminal of the op-amp. We can compute the node voltage using NVA and the fact that no current will flow into the op-amp.

Applying KCL at the "+" terminal of the comparator, and because there is no current entering the $+$ terminal of the comparator we get:

$$I_{R_1} = I_{R_2}\,.$$

Substituting the currents using Ohm's law we get:

$$\frac{V_{\text{ref}} - u_+}{R_1} = \frac{u_+ - V_{\text{out}}}{R_2} \rightarrow u_+ = \frac{R_2}{R_1 + R_2} V_{\text{ref}} + \frac{R_1}{R_1 + R_2} V_{\text{out}}\,.$$

7. Least Squares for Robotics (16 points)

Robots rely on sensors for understanding their environment and navigating in the real world. These sensors must be calibrated to ensure accurate measurements, which we explore in this problem.

(a) (3 points) Your robot is equipped with two forward-facing sensors – a radar and camera.

However, the sensors are placed with an offset (i.e. a gap) of ℓ in meters (m), as depicted in Fig. 23, and you want to find its value. The radar returns a range ρ in meters (m) and heading angle θ in radians (rad) with respect to the object. In contrast, the camera only returns an angle, ϕ in radians (rad), with respect to the object.

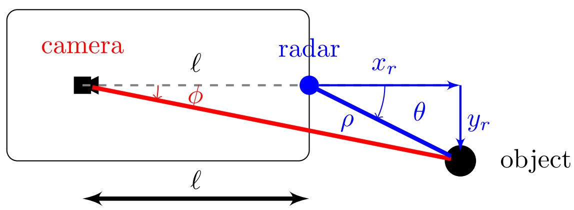


Figure 23: Sensor Placement and Offset ℓ.

These relationships are summarized by the following sensor model, where x_r and y_r are the Cartesian coordinates of the object with respect to the radar:

$$x_r = \rho\cos(\theta)\,, \tag{7}$$

$$y_r = \rho\sin(\theta)\,, \tag{8}$$

$$\tan(\phi) = \frac{y_r}{x_r+\ell}\,. \tag{9}$$

Assuming $\phi \neq 0$, use equations (7), (8), (9) to express ℓ in terms of ρ, θ, and ϕ.

Solutions:

From the sensor model, we have:

$$(x_r+\ell)\tan(\phi) = y_r$$
$$\Rightarrow \ell = \frac{y_r}{\tan(\phi)} - x_r$$
$$\Rightarrow \ell = \rho\left(\frac{\sin(\theta)}{\tan(\phi)} - \cos(\theta)\right).$$

Note: We stipulate that $\phi \neq 0$ since otherwise division by $\tan(\phi)$ would not be well-defined. When $\phi = 0$, the object would be located right in front of both the radar and camera, and any positive value of ℓ would solve the system of equations. This explanation is not required for full credit.

(b) (5 points) Often it is difficult to precisely identify the value of ℓ. To learn the value of ℓ you decide to take a series of measurements. In particular, you take N measurements and get the equations:

$$a\,\ell + e_i = b_i$$

for $1 \leq i \leq N$. Here $a \neq 0$ is a fixed and known constant. Each b_i represents your i^{th} measurement and e_i represents the error in your measurement. While you know all of the b_i values, you do not know the error values e_i.

We can write this equation in a vector format as:

$$\mathbf{A}\ell + \vec{e} = \vec{b},$$

where $\mathbf{A} = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}$, $\vec{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$.

In this simple 1-D case, the least squares solution is a scaled version of the average of $\{b_i\}_{i=1}^{N}$.

Find the best estimate for ℓ, denoted as $\hat{\ell}$, using least squares. Simplify your expression and express $\hat{\ell}$ in terms of a, b_i, and N. Your answer may not include any vector notation.

Note: A is a vector and not a matrix.

Solutions:

$\hat{\ell}$ is given by the least square solution:

$$\begin{aligned}\hat{\ell} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \\ &= (Na^2)^{-1} a \sum_{i=1}^{N} b_i \\ &= \frac{\sum_{i=1}^{N} b_i}{aN}.\end{aligned}$$

(c) (8 points) Now we turn to the task of controlling the robot's velocity and acceleration, which is a key requirement for navigation.

We use the following model for the robot, which describes how the velocity and acceleration of the robot changes with timestep k:

$$\begin{bmatrix} v[k+1] \\ a[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v[k] \\ a[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[k],$$

where

- k is the timestep;
- $v[k]$ is the velocity state at timestep k;
- $a[k]$ is the acceleration state at timestep k;
- $j[k]$ is the jerk (derivative of acceleration) control input at timestep k.

We start at a known initial state $\begin{bmatrix} v[0] \\ a[0] \end{bmatrix}$, and we want to find $j[0]$ to set $\begin{bmatrix} v[1] \\ a[1] \end{bmatrix}$ as close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as possible. For this, we minimize:

$$E = \left\| \begin{bmatrix} v[1] \\ a[1] \end{bmatrix} \right\|^2.$$

Find the best estimate for the optimal choice of jerk, $\hat{j}[0]$, by using least squares method to minimize E. Express your solution in terms of $v[0]$ and $a[0]$. Show your work.

Hint: Rewrite E in terms of $j[0]$ and other relevant terms.

Solutions:

Starting from the hint, we try to rewrite the cost E. Applying the dynamics model, we find that:

$$\begin{aligned} E &= \left\| \begin{bmatrix} v[0]+a[0] \\ a[0]+j[0] \end{bmatrix} \right\|^2 \\ &= \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[0] - \begin{bmatrix} -v[0]-a[0] \\ -a[0] \end{bmatrix} \right\|^2 \\ &= \left\| \mathbf{A} j[0] - \vec{b} \right\|^2. \end{aligned}$$

Therefore, $\hat{j[0]}$ is given by the least square solution:

$$\begin{aligned} \hat{j[0]} &= (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b} \\ &= (1)^{-1} \times \ (-a[0]) \\ &= -a[0]. \end{aligned}$$

8. Proof (10 points)

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. The eigenvalues and eigenvectors of $\mathbf{A}$ are given by $(\alpha_1, \vec{v}_1)$, $(\alpha_2, \vec{v}_2)$, $\cdots$, $(\alpha_n, \vec{v}_n)$, where all the α_i, $1 \le i \le n$, are distinct. Similarly the eigenvalues and eigenvectors of $\mathbf{B}$ are given by $(\beta_1, \vec{v}_1)$, $(\beta_2, \vec{v}_2)$, $\cdots$, $(\beta_n, \vec{v}_n)$, where all the β_i, $1 \le i \le n$, are distinct.
NOTE: $\mathbf{A}, \mathbf{B}$ *have identical eigenvectors.*

Prove that:

$$\mathbf{A}\,\mathbf{B}\,\vec{x} \;=\; \mathbf{B}\,\mathbf{A}\,\vec{x},$$

for any vector $\vec{x} \in \mathbb{R}^n$.

Solutions:
To prove that the matrices $\mathbf{A}$ and $\mathbf{B}$ commute for any $\vec{x} \in \mathbb{R}^n$, we must first be able to write any such $\vec{x}$ in terms of the shared matrix eigenvectors $\vec{x} = \sum_{j=1}^{n} c_j\, \vec{v}_j$. We know that this is true from the theorems proved in lecture and the notes, since $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ forms a basis for $\mathbb{R}^n$. You must acknowledge this to receive full credit.

Since eigenvectors $\vec{v}_j$ form a basis of $\mathbb{R}^n$, we can write any vector as a linear combination of the eigenvectors, i.e. $\vec{x} = \sum_{j=1}^{n} c_j \vec{v}_j$.

We know that any $\vec{x} \in \mathbb{R}^n$ can be uniquely expressed in the identical basis of eigenvectors for $\mathbf{A}$ and $\mathbf{B}$.

$$\begin{aligned}
\mathbf{A}\,\mathbf{B}\,\vec{x} &= \sum_{j=1}^{n} c_j\, \mathbf{A}\,\mathbf{B}\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\, \mathbf{A}\,\beta_j\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\, \beta_j\,\mathbf{A}\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\, \beta_j\,\alpha_j\,\vec{v}_j.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{B}\,\mathbf{A}\,\vec{x} &= \sum_{j=1}^{n} c_j\,\mathbf{B}\,\mathbf{A}\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\,\mathbf{B}\,\alpha_j\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\,\alpha_j\,\mathbf{B}\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\,\alpha_j\,\beta_j\,\vec{v}_j \\
&= \sum_{j=1}^{n} c_j\,\beta_j\,\alpha_j\,\vec{v}_j\,.
\end{aligned}$$

Therefore bot the expressions are equal. The key property we are exploiting is that each $\vec{v}_j$ is simultaneously an eigenvector of $\mathbf{A}$ and $\mathbf{B}$.
The only other property we needed was matrix linearity $\mathbf{A}(a\,\vec{x} + b\,\vec{y}\,) = a\,\mathbf{A}\,\vec{x} + b\,\mathbf{A}\,\vec{y}$. This concludes the proof.

In case you want a reminder on how to show that $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ forms a basis for $\mathbb{R}^n$:
It was shown in lecture that the eigenvectors of a matrix with entirely distinct eigenvalues are all mutually linearly independent and thus form a basis for $\mathbb{R}^n$.

- **i. Fundamental Idea:** Since each $\vec{v}_j$ lives in $\mathbb{R}^n$ and there are n such vectors, the set of eigenvectors will form a basis of $\mathbb{R}^n$ *if and only if* they are linearly independent.
- **ii. Prepare contradiction:** Suppose the converse; the eigenvectors are linearly dependent so there is a set of constants d_j we can choose such that $\sum_{j=1}^{n} d_j \vec{v}_j = \vec{0}$. Now we know multiplying $\vec{0}$ by any matrix will still result in zero, and furthermore any inner product $\left\langle \vec{a}\ ,\ \vec{0} \right\rangle$ will also be zero (regardless of $\vec{a}$).
- **iii. Arriving at the paradox:** Without loss of generality, say α_n is the eigenvalue of $\mathbf{A}$ with the greatest absolute value $|\alpha_n| > |\alpha_j|$ for any $j \in 1, 2, \ldots, n-1$.
  NOTE: this is a strong inequality since the eigenvalues of $\mathbf{A}$ *are all distinct!*
  Next compute the inner product of the expression $\left\langle \vec{v}_n\ ,\ \frac{1}{\alpha_n^N}\mathbf{A}^N\vec{0} \right\rangle$ (which must always be zero) for any positive integer N.

$$0 \quad \equiv \quad \left\langle \vec{v}_n\ ,\ \frac{1}{\alpha_n^N}\mathbf{A}^N\vec{0} \right\rangle \quad = \quad \sum_{j=1}^{n} \frac{d_j}{\alpha_n^N}\left\langle \vec{v}_n\ ,\ \mathbf{A}^N\vec{v}_j \right\rangle \quad = \quad \sum_{j=1}^{n} d_j \left(\frac{\alpha_j}{\alpha_n}\right)^N \left\langle \vec{v}_n\ ,\ \vec{v}_j \right\rangle.$$

In the large N limit, the term in parentheses will vanish for all terms except for $j = n$. The only remaining term is $(\alpha_n/\alpha_n)^N = 1$. Thus we arrive at the final contradiction

$$0 \quad \equiv \quad d_n\,|\vec{v}_n|^2 \quad > \quad 0. \qquad \square$$

This means there is no choice of d_j such that $\sum_{j=1}^{n} d_j \vec{v}_j = \vec{0}$.
Q.E.D.

Small note:
In the event that our linear combination happened to have $d_n = 0$, then we can return to step 'ii.' with $n-1$ in place of n (since $d_n = 0$). While it could be that $d_{n-1} = 0$ as well, the procedure here can be continuously applied until you reach a nonzero d_j. Further, there *must* be a nonzero d_j as required by the very definition of linear dependence!!