Physics 7C Midterm Solutions

Problem 1:

(a) 5 points: Place the object at infinity and the image at $W - \Delta x$, both relative to the eye's lens.

$$\frac{1}{\infty} + \frac{1}{W - \Delta x} = \frac{1}{f_0} \quad (1)$$
$$f_0 = W - \Delta x \quad (2)$$

(b) 4 points: The eye's lens refracts the light rays too much. Thus, the lens on the glasses must be diverging, so counter the over-convergence caused by the eyes. Nearsighted people need diverging lenses.

(c) 12 points: An object at infinity in front of the glasses lens produces a virtual image from the glasses lens. This image, which is at a distance $|f_1|$ in front of the glasses lens (on the same side as the object), is the object of the eye lens. This virtual object is $L + |f_1|$ in front of the eye lens. A real image of that virtual object is produced on the retina by the eye lens, i.e., a distance W on the other side of the eye lens. Thus, using the lens equation of the eye lens,

$$\frac{1}{L + |f_1|} + \frac{1}{W} = \frac{1}{f_0} = \frac{1}{W - \Delta x} \quad (3)$$
$$|f_1| = W\left(\frac{W}{\Delta x} - 1\right) - L \quad (4)$$
$$f_1 = L - W\left(\frac{W}{\Delta x} - 1\right) \quad (5)$$

Note that f_1 is negative, because the glasses lens is diverging.

(d) 4 points: Diverging lenses have negative focal lengths, which means their image distances are always negative if the object distance is positive. That is, the image of a diverging lens is on the same side as a positive real object, even though the actual rays refract through the lens instead of reflect back. Thus, Maria sees a virtual image.

Problem 2:

(a) 15 points: First, apply Snell's law between two adjacent layers. Then Taylor expand $\sin(\theta + \Delta\theta)$ around θ.

$$n \sin\theta = (n + \Delta n)\sin(\theta + \Delta\theta) \quad (6)$$
$$n \sin\theta = (n + \Delta n)[\sin\theta + (\cos\theta)\Delta\theta + \ldots]) \quad (7)$$
$$n \sin\theta = n\sin\theta + \Delta n \sin\theta + n\Delta\theta\cos\theta \quad (8)$$

We've only kept terms up to linear order in the small quantities Δn and $\Delta\theta$. After simplifying further, we find

$$0 = \Delta n \sin\theta + n\Delta\theta\cos\theta \quad (9)$$

Now we recognize from the chain rule that $(\cos\theta)\Delta\theta = \Delta(\sin\theta)$. We can substitute this into our equation and recognize the product rule.

$$0 = (\Delta n)\sin\theta + n\Delta(\sin\theta) \tag{10}$$
$$0 = \Delta(n\sin\theta) \tag{11}$$

Thus, the familiar $n\sin\theta$ is our conserved quantity between any pair of thin layers, no matter how far apart they are.

(b) 5 points: Recall that $n(0) = 1$. The relationship between the top and bottom of the atmosphere can be found by applying the conserved quantity.

$$n(D)\sin(\theta(D)) = \sin(\theta(0)) \tag{12}$$
$$\theta(D) = \arcsin\left(e^{-D/\Delta d}\sin(\theta(0))\right) \tag{13}$$

(c) 5 points: In the limit $\Delta d \ll D$, we have $e^{-D/\Delta d} \to 0$. Since $\arcsin(0) = 0$, we find $\theta(D) = 0$. The sketch of the ray enters the atmosphere at $\pi/4$, then curves through the atmosphere until it exits at normal incidence.

Problem 3:

(a) 4 points: Recall that for electromagnetic waves with fields $\mathbf{E}$ and $\mathbf{B} = \mathbf{E}/c$, the energy density stored in the fields is given by

$$u_{tot} = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2 = \epsilon_0 E^2 \tag{14}$$

Thus, for waves traveling at speed c, the power flux (energy per area per second) is

$$S = c \cdot u_{tot} = c\epsilon_0 E^2. \tag{15}$$

Without any loss of generality, let's assume the detector is located at position $x = 0$, so that, at the detector, we have

$$\mathbf{E}_1 = E_0\cos(-\omega t) = E_0\cos(\omega t) \tag{16}$$

Then, the power flux at the detector is

$$S = c\epsilon_0 E_1^2\cos^2(\omega t) \tag{17}$$

The intensity I_1 is then given by the average of the power flux over a period of the wave:

$$I_1 = \langle S\rangle = c\epsilon_0 E_0^2 \cdot \langle\cos^2(\omega t)\rangle \tag{18}$$

where $\langle\cdot\rangle$ denotes an average over the period. We can either just remember that the average of $\cos^2$ is $\frac{1}{2}$, or we can show explicitly:

$$\frac{1}{T}\int_0^T \cos^2(\omega t)dt = \frac{2\pi}{\omega}\int_0^{2\pi/\omega}\cos^2(\omega t)dt = \frac{1}{2\pi}\int_0^{2\pi}\cos^2 x dx \tag{19}$$

where $x = \omega t \Rightarrow dt = dx/\omega$. Using $\cos^2(x) = (1 + \cos(2x))/2$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}(1+\cos(2x))dx = \frac{1}{2\pi}\cdot\frac{1}{2}\cdot x\Big|_0^{2\pi} + \frac{1}{2\pi}\cdot\frac{1}{2}\cdot\sin(2x)\Big|_0^{2\pi} = \frac{1}{2} \tag{20}$$

In any case, we have that

$$I_1 = \frac{1}{2}c\epsilon_0 E_0^2 \tag{21}$$

(b) 7 points: If both waves are present and $\phi = 0$, then at the detector ($x = 0$) we have

$$\begin{aligned}\mathbf{E}_{tot} &= E_0\hat{\mathbf{y}}\cos(-\omega t) + E_0\hat{\mathbf{y}}\cos(-\omega t) = 2E_0\hat{\mathbf{y}}\cos(-\omega t)\\ &= 2E_0\hat{\mathbf{y}}\cos(\omega t)\end{aligned} \tag{22}$$

and hence

$$I_{tot}(\phi = 0) = \frac{1}{2}c\epsilon_0(2E_0)^2 = 4I_1 \tag{23}$$

$$\frac{I_{tot}(0)}{I_1} = 4 \tag{24}$$

(c) 6 points: If $\phi = \pi$, then at $x = 0$ the total field is

$$\begin{aligned}\mathbf{E}_{tot} &= E_0\hat{\mathbf{y}}\cos(-\omega t) + E_0\hat{\mathbf{y}}\cos(-\omega t + \pi)\\ &= E_0\hat{\mathbf{y}}\cos(-\omega t) - E_0\hat{\mathbf{y}}\cos(-\omega t)\\ &= 0\end{aligned} \tag{25}$$

where in the second line we have used the fact that $\cos(x + \pi) = -\cos(x)$, for any x. Therefore,

$$\frac{I_{tot}(\pi)}{I_1} = 0 \tag{26}$$

(d) 8 points: Recall the identity

$$\cos(x) + \cos(y) = 2\cos\big(\frac{x+y}{2}\big)\cos\big(\frac{x-y}{2}\big) \tag{27}$$

Using this, with $\phi = \frac{\pi}{3}$, the total electric field at the detector becomes

$$\begin{aligned}\mathbf{E}_{tot} &= E_0\hat{\mathbf{y}}\cos(-\omega t) + E_0\hat{\mathbf{y}}\cos(-\omega t + \pi/3)\\ &= 2E_0\hat{\mathbf{y}}\cos(-\omega t + \pi/6)\cos(\pi/6)\\ &= \sqrt{3}E_0\hat{\mathbf{y}}\cos(\omega t - \pi/6)\end{aligned} \tag{28}$$

Here, we used $\cos(\pi/6) = \sqrt{3}/2$. Therefore, the intensity is

$$I(\pi/3) = c\epsilon_0(\sqrt{3}E_0)^2\langle\cos^2(\omega t + \pi/6)\rangle = 3c\epsilon_0 E_0^2 \cdot \frac{1}{2} = 3I_1 \tag{29}$$

where we have used the fact that adding a constant to the argument of the cosine function doesn't change its average value. Therefore,

$$\frac{I_{tot}(\pi/3)}{I_1} = 3 \tag{30}$$

Alternatively, we note that

$$\cos(-\omega t) + \cos(-\omega t + \pi/3) = \mathrm{Re}\left[e^{-i\omega t} + e^{i(-\omega t + \pi/3)}\right] \tag{31}$$

$$= \mathrm{Re}\left[e^{-i\omega t}\Big(1 + e^{i\pi/3}\Big)\right] \tag{32}$$

Then factor out $e^{i\pi/6}$ to get the sum of $e^{i\pi/6} + e^{-i\pi/6}$, which is $2\cos(\pi/6)$, etc.

Problem 4:

(a) 7 points: Let us label the event of the rocket launch as *event A*, and the event of the booster ejection as *event B*. Then A and B occur at the same place in the Rocket frame. Thus, the time between the events in the rocket frame, τ_1, is the *proper time* between the events. Therefore, the dilated time measured by the earthbound observer is given via the time dilation equation:

$$t_1 = \gamma \tau_1 = \frac{1}{\sqrt{1-\beta^2}} \tau_1 \tag{33}$$

Since the rocket travels at constant speed βc with respect to the Earth, the (Earth) distance traveled in time t_1 is

$$L_1 = \beta c \cdot t_1 = \frac{\beta c}{\sqrt{1-\beta^2}} \tau_1 \tag{34}$$

(b) 11 points: The rocket speed, as measured on Earth, is $v = \beta c$. Label the booster speed, as measured on Earth by u, and its speed as measured by the rocket (given) as $u' = -\beta' c$. We use the minus sign because the booster is moving towards the Earth. Then, from the velocity addition formula,

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \Longrightarrow u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \tag{35}$$

we find that the speed of the booster in the Earth frame is

$$u = \frac{-\beta' c + \beta c}{1 - \beta' \beta} = \frac{c(\beta - \beta')}{1 - \beta \beta'} \tag{36}$$

Then, since the booster is ejected a distance L_1 from the Earth, it takes an amount of (Earth) time

$$\Delta t = \frac{L_1}{|u|} = \left[\frac{\beta c}{\sqrt{1-\beta^2}} \tau_1 \right] \cdot \left[\frac{1 - \beta \beta'}{c(\beta' - \beta)} \right] \tag{37}$$

to reach the Earth. Using the expression for t_1 from part (a), the total time of the journey is

$$\begin{aligned} t_2 = t_1 + \Delta t &= \frac{1}{\sqrt{1-\beta^2}} \tau_1 + \left[\frac{\beta c}{\sqrt{1-\beta^2}} \tau_1 \right] \cdot \left[\frac{1 - \beta \beta'}{c(\beta' - \beta)} \right] \\ &= \frac{\tau_1}{\sqrt{1-\beta^2}} \left[\frac{c(\beta' - \beta)}{c(\beta' - \beta)} + \frac{\beta c (1 - \beta \beta')}{c(\beta' - \beta)} \right] \\ &= \frac{\tau_1}{\sqrt{1-\beta^2}} \left[\frac{\beta' - \beta^2 \beta'}{\beta' - \beta} \right] \\ &= \frac{\tau_1 \beta'}{\sqrt{1-\beta^2}} \frac{1 - \beta^2}{\beta' - \beta} \\ &= \tau_1 \frac{\beta'}{\beta' - \beta} \sqrt{1-\beta^2} \end{aligned} \tag{38}$$

(c) 7 points: Label again the rocket-launch by *event A*, and the booster arriving at Earth by *event C*. Then A and C occur at the same place in the Earth frame. Thus, the Earth time between the two events, t_2 is the *proper* time between the two events. To find the dilated time in the rocket frame moving at speed βc, we use the time dilation equation:

$$\begin{aligned} \tau_2 = \gamma t_2 &= \frac{1}{\sqrt{1-\beta^2}} \cdot \tau_1 \frac{\beta'}{\beta' - \beta} \sqrt{1-\beta^2} \\ &= \tau_1 \cdot \frac{\beta'}{\beta' - \beta} \end{aligned} \tag{39}$$

<u>Note</u>: In this case we used $\tau_2 = \gamma t_2$, whereas in part (a) we used $t_1 = \gamma \tau_1$.