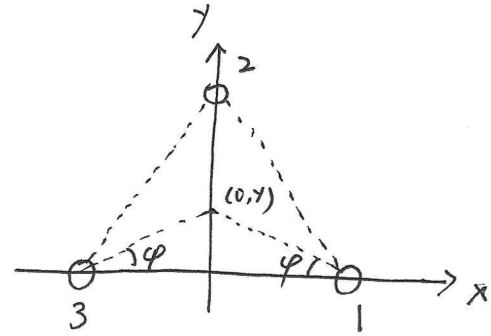
a) Let $\theta = 60^\circ$



The force from star 2 to star 1 is

$$\vec{F}_{21} = \frac{GM^2}{4b^2}(-\cos\theta\hat{x} + \sin\theta\hat{y})$$

The force from star 3 to star 1 is

$$\vec{F}_{31} = -\frac{GM^2}{4b^2}\hat{x}$$

Thus the y component of the total gravitational force on star 1 is

$$F_{1,y} = F_{21,y} + F_{31,y} = \frac{\sqrt{3}GM^2}{8b^2}$$

b) the y component of the total gravitational force on star ~~2~~ 1 is

$$F_{1,x} = F_{21,x} + F_{31,x} = -\frac{3GM^2}{8b^2}$$

c) The gravitational field is

$$\vec{g} = \vec{g}_1 + \vec{g}_2 + \vec{g}_3$$

$$= \frac{GM}{b^2+Y^2}(+\cos\varphi\hat{x} - \sin\varphi\hat{y}) + \frac{GM}{(\sqrt{3}b-Y)^2}\hat{y} + \frac{GM}{b^2+Y^2}(-\cos\varphi\hat{x} - \sin\varphi\hat{y})$$

$$= \frac{GM}{(\sqrt{3}b-Y)^2}\hat{y} - \frac{2GM\sin\varphi}{b^2+Y^2}\hat{y}$$

$$= \left(\frac{GM}{(\sqrt{3}b-Y)^2} - \frac{2GMY}{(b^2+Y^2)^{3/2}}\right)\hat{y}$$

d) From the center of the triangle to the origin, $d\vec{l} = \cancel{dy} - dY$

$$W = \int m\vec{g}\cdot d\vec{l} = \int_0^{\frac{\sqrt{3}b}{3}}\left(-\frac{GM}{(\sqrt{3}b-Y)^2} + \frac{2GMY}{(b^2+Y^2)^{3/2}}\right)dY$$

e) The intial location is $(0, \frac{\sqrt{3}b}{3})$, the final location is $(0,0)$

$$W = E_{initial} - E_{final}$$

$$= -\frac{3GMm}{2\sqrt{3}b/3} + \frac{2GMm}{b} + \frac{GMm}{\sqrt{3}b}$$

$$= \left(2 - \frac{7\sqrt{3}}{6}\right)\frac{GMm}{b}$$

Jet Pack Skiing

1. Assuming that the motion is frictionless, the work done by the jetpack is equal to the change of the mechanical energy:

$$\boxed{W_{Jetpack} = E_{M,top} - E_{M,start} = mgR + \frac{1}{2}mv_{top}^2}$$

2. The normal force is perpendicular to the movement: $\vec{N} \perp \vec{dl}$. Therefore:

$$\boxed{W_N = \int_{l=0}^{l=\pi R/2} \vec{N}\vec{dl} = 0}$$

3. We can solve this question using 2 methods:

   a. **Using energy**: Let's consider the movement between 0 and s ($s < \pi R/2$). The change of the mechanical energy between 0 and s is equal to the work done by the force of the jetpack F. Let θ_s be the angle at position s. Therefore we have:

$$W_{Jetpack} = \int_0^s \vec{F}(s)\vec{dl} = \underbrace{\int_0^s F(s)ds}_{\text{because F is parallel to dl}} = mgR\sin\theta_s + \frac{1}{2}mv^2(s)$$

   Now, given that $\theta_s = \frac{s}{R}$ and $v = qs$ we have:

$$\int_0^s F(s)ds = mgR\sin\frac{s}{R} + \frac{1}{2}mq^2s^2$$

   By differentiating the above equation with respect to s we obtain:

$$\boxed{F(s) = mg\cos\frac{s}{R} + mq^2 s}$$

   b. **Using Newton's 2nd law:** Netwon's second law:

$$m\vec{a} = \vec{F} + \vec{N} + \vec{F_g}$$

   We project this equation on the direction of the movement, we obtain at a position s:

$$ma(s) = m\frac{dv}{dt}(s) = F(s) - mg\cos\theta_s$$

   Given that $v = qs$, $v = \frac{ds}{dt}$ and $\theta_s = s/R$ we get:

$$m\frac{dv}{dt} = mq\frac{ds}{dt} = mqv = mq^2s = F(s) - mg\cos\frac{s}{R}$$

   Finally:

$$\boxed{F(s) = mg\cos\frac{s}{R} + mq^2 s}$$

4. Between the starting position and the top the work is done by the force of the Jetpack F and gravity. Given that $F = F_0$ for the first half and 0 in the second half, and that the direction of $\vec{F}$ is parallel to the movement, we have:

$$W_F = F_0\frac{\pi R}{4} \quad ; \quad W_{F_g} = -mgR$$

   Therefore using the work-energy principle:

$$F_0\frac{\pi R}{4} - mgR = \frac{1}{2}mv_{top}^2 \implies \boxed{F_0 = \frac{2mv_{top}^2}{\pi R} + \frac{4mg}{\pi}}$$

5. • The vertical component of the force should overcome gravity. Therefore:

$$\boxed{F_0 > \sqrt{2}mg}$$

• To determine the corresponding V_{top} we will use the work-energy principle. $\vec{F} = \vec{F_0} = F_0(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j})$ is constant. Therefore:

$$W_F = \int_0^{\pi R/2} \vec{F_0}\vec{dl} = R\int_0^{\pi/2} \vec{F_0}(\sin\theta\vec{i} + \cos\theta\vec{j})d\theta = \frac{RF_0}{\sqrt{2}} \underbrace{\int_0^{\pi/2} (\sin\theta + \cos\theta)d\theta}_{=2}$$

Therefore:

$$W_F = \sqrt{2}RF_0$$

The minimum value of F_0 gives: $W_{F,min} = 2mgR$. Using the work-energy principle we obtrain:

$$\frac{1}{2}mv_{top,min}^2 = 2mgR - mgR = mgR$$

Finally:

$$\boxed{v_{top,min} = \sqrt{2gR}}$$

3. Sliding Masses [25 points]

Two particles move on a horizontal frictionless surface. Particle 1 has a mass m_1 and initial speed v_1, moving in the $+x$ direction. Particle 2 has a mass m_2 and initial speed v_2, moving in the $(\hat{\mathbf{i}} + \hat{\mathbf{j}})/\sqrt{2}$ direction.

(a) [5 points] If the particles collide and stick together, what are the x and y components of the velocity vector after the collision?

(b) [5 points] Calculate the final total kinetic energy minus the initial total kinetic energy.

(c) [5 points] Calculate the change in the momentum vector of Particle 1 (that is, final minus initial). Your answer should state the x and y components of this vector.

(d) [5 points] For this sub-problem, we assume $m_1 = m_2$. What are the initial velocity vectors of each particle *relative to the center of mass*?

This is a completely separate problem. We study a collision in 1D. A soccer ball (mass $= m_s$) has initial velocity v_0. A very massive bowling ball (mass $= m_b$) is initially at rest. After the collision, the total kinetic energy of the system is 5/6 of its original value.

(e) [5 points] Calculate v_b, the velocity of the bowling ball after the collision.

a-) $\vec{v}_1 = v_1 \hat{i}$ $\quad \vec{v}_2 = \frac{v_2}{\sqrt{2}}\hat{i} + \frac{v_2}{\sqrt{2}}\hat{j}$

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}_f$$

$$\left(m_1 v_1 + m_2\frac{v_2}{\sqrt{2}}\right)\hat{i} + \frac{m_2 v_2}{\sqrt{2}}\hat{j} = (m_1+m_2)\vec{v}_f$$

$$\vec{v}_f = \left(\frac{m_1}{m_1+m_2}v_1 + \frac{m_2}{m_1+m_2}\frac{v_2}{\sqrt{2}}\right)\hat{i} + \frac{m_2}{m_1+m_2}\frac{v_2}{\sqrt{2}}\hat{j}$$

b-) $$\frac{1}{2}(m_1+m_2)v_f^2 - \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_2 v_2^2$$

$$= \frac{1}{2}(m_1+m_2)\frac{m_1^2 v_1^2 + \sqrt{2}\, m_1 m_2 v_1 v_2 + m_2^2 v_2^2}{(m_1+m_2)^2} - \frac{1}{2}\frac{m_1(m_1+m_2)}{m_1+m_2}v_1^2 - \frac{1}{2}\frac{m_2(m_1+m_2)}{m_1+m_2}v_2^2$$

$$= -\frac{m_1 m_2\left(v_1^2 - \sqrt{2}\, v_1 v_2 + v_2^2\right)}{2(m_1+m_2)}$$

c-) $\Delta\vec{p}_1 = m_1\vec{v}_{1f} - m_1\vec{v}_{1i}$

$$= m_1\left[\left(\frac{m_1}{m_1+m_2}v_1 + \frac{m_2}{m_1+m_2}\frac{v_2}{\sqrt{2}}\right)\hat{i} + \left(\frac{m_2}{m_1+m_2}\frac{v_2}{\sqrt{2}}\right)\hat{j}\right] - m_1 v_1\hat{i}$$

$$= m_1\left[\left\{v_1\left(\frac{m_1}{m_1+m_2} - 1\right) + \frac{m_2 v_2}{m_1+m_2}\cdot\frac{1}{\sqrt{2}}\right\}\hat{i} + \frac{m_2}{m_1+m_2}\frac{v_2}{\sqrt{2}}\hat{j}\right]$$

$$= \frac{m_1(2m_1v_1 + \sqrt{2}m_2v_2)}{2(m_1+m_2)}\hat{i} + \frac{m_1m_2v_2}{\sqrt{2}(m_1+m_2)}$$

d-) $m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1+m_2)\vec{v}_{cm}$

but this shows $\vec{v}_{cm} = \vec{v}_f$ in previous problem.

If $m_1 = m_2 = m$, then $\vec{v}_{cm} = \vec{v}_{f_{m_1=m_2=m}} =$

$$\frac{v_2}{2\sqrt{2}}\hat{j} + \frac{2v_1 + \sqrt{2}v_2}{4}\hat{i}$$

velocity w.r.t CM $= \vec{v}_{wrtcm} = \vec{v} - \vec{v}_{cm}$

then $\vec{v}_{1cm} = \vec{v}_1 - \vec{v}_{cm} = \frac{2v_1 - \sqrt{2}v_2}{4}\hat{i} - \frac{v_2}{2\sqrt{2}}\hat{j}$

$\vec{v}_{2cm} = \vec{v}_2 - \vec{v}_{cm} = \frac{-2v_1 + \sqrt{2}v_2}{4}\hat{i} + \frac{v_2}{2\sqrt{2}}\hat{j}$

$\vec{v}_{1cm} + \vec{v}_{2cm} = 0$ as expected.

e-) $m_1 v_{1i} + \cancel{m_2 v_{2i}}^{0} = m_1 v_{1f} + m_2 v_{2F}$

$$\frac{1}{2} m_1 v_{1i}^2 + \cancel{\frac{1}{2} m_2 v_{2i}^2}^{0} = \frac{5}{6}\left(\frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2\right)$$

$v_{1i} = v_0, \quad v_{2i} = 0$ then $v_{2f} = v_b$

$$m_1 v_0 = m_1 v_{1f} + m_2 v_{2f} \quad (1)$$

$$\frac{1}{2} m_1 v_0^2 = \frac{5}{6}\left(\frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2\right) \quad (2)$$

From (1): $v_{1f} = \dfrac{m_1 v_0 - m_2 v_{2f}}{m_1}$. Substitute in (2)

$$\frac{5}{6}\frac{1}{2} m_1 v_0^2 = \left(\frac{1}{2} m_1 \left(\frac{m_1 v_0 - m_2 v_{2f}}{m_1}\right)^2 + \frac{1}{2} m_2 v_{2f}^2\right)$$

1 eq'n 1 unknown. The roots are:

$$\frac{6 m_1 m_2 v_0 \pm \sqrt{6}\, m_1 \sqrt{m_2 (m_1 + 5 m_2) v_0^2}}{6 m_1 m_2 + 6 m_2^2} = v_{2f} = v_b$$

Plus root is positive, minus one is negative. We expect bowling ball to move in the +x direction thus the answer is the positive one.

Sloped Spring

a)

$$k(D_0 - D) = mgsin(\theta)$$

$$\boxed{k = \frac{mgsin(\theta)}{D_0 - D}}$$

b)

$$\frac{k}{2}(D_0 - D)^2 = mgxsin(\theta) + \frac{k}{2}(D_0 - D - x)^2$$

$$x = D_0 - D$$

$$\frac{k}{2}(D_0 - D) = mgsin(\theta)$$

$$\boxed{k = 2\frac{mgsin(\theta)}{D_0 - D}}$$

c)

$$T(l) = \frac{k}{2}(D_0 - D)^2 - mglsin(\theta) - \frac{k}{2}(D_0 - D - l)^2$$

$$\frac{dT}{dl} = k(D_0 - D - l) - mgsin(\theta)$$

$$\left.\frac{dT}{dl}\right|_{l_{max}} = 0 = k(D_0 - D - l_{max}) - mgsin(\theta)$$

$$l_{max} = D_0 - D - \frac{mgsin(\theta)}{k}$$

$$\boxed{l_{max} = \frac{D_0 - D}{2}}$$

$$\frac{m}{2}v_{max}^2 = T(l_{max})$$

$$T(l_{max}) = \frac{mgsin(\theta)}{4}(D_0 - D)$$

$$v_{max} = \sqrt{\frac{2T(l_{max})}{m}}$$

$$\boxed{v_{max} = \sqrt{\frac{gsin(\theta)}{2}(D_0 - D)}}$$

d)

$$W_{spring} = \int_{D_0-D}^{D_0-D-l_{max}} -kx\,dx$$

$$W_{spring} = \frac{k}{2}((D_0 - D)^2 - (D_0 - D - l_{max})^2)$$

$$\boxed{W_{spring} = \frac{3k}{4}mgsin(\theta)(D_0 - D)}$$

e)

From symmetry,

$$\boxed{y_{cm} = 0}$$

$$x_{cm} = \frac{\int \frac{M}{\pi R} x\, dx}{M}$$

$$x_{cm} = \frac{\int_{-\pi/2}^{\pi/2} R^2 cos(\theta)\, d\theta}{\pi R}$$

$$\boxed{x_{cm} = \frac{2R}{\pi}}$$