EECS 16A Designing Information Devices and Systems I
Spring 2021 Midterm 1

Midterm 1 Solution

1. **HONOR CODE**
If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my one reference cheat sheet.*
- *I did not collaborate with any other human being on this exam.*

2. (a) **Tell us about something that makes you happy.** *All answers will be awarded full credit; you can be brief.* **(2 Points)**

(b) **What is one of your hobbies?** *All answers will be awarded full credit; you can be brief.* **(2 Points)**

3. Mechanical Linear Algebra (22 points)

(a) (4 points) Consider the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + x_3 = 2 \\ -2x_1 + 2x_2 - 2x_3 = -4 \\ 2x_1 + 2x_3 = 3 \end{cases}$$

Write the system of equations in augmented matrix form and bring to **reduced row echelon form** through Gaussian elimination. **How many solutions (if any) does this system of equations have?**

Solution: The augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ -2 & 2 & -2 & -4 \\ 2 & 0 & 2 & 3 \end{array}\right]$$

To get to Reduced Row echelon form, we use Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ -2 & 2 & -2 & -4 \\ 2 & 0 & 2 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 3 \end{array}\right] \xrightarrow{R_4 = R_4 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_4 = R_4 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = R_1 - R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system of equations has infinite solutions.

(b) (2 points) For the new row reduced matrix below:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & d-5 \end{array}\right]$$

For what value of d (if any) will the system have a solution? If it does have a solution, what is the solution?

Solution: We fist reduce the matrix futher:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & d-5 \end{array}\right] \xrightarrow{R_1=R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & d-5 \end{array}\right]$$

For the system of equations to be consistent (have at least one solution), the last row has to be identically 0. Therefore, $d = 5$.
When $d = 5$,

$$\begin{cases} x_1 = -2 \\ x_2 = -2/3 \\ x_3 = 5 \end{cases} .$$

(c) (6 points) For vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ shown below:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4.5 \\ 6 \\ 1 \end{bmatrix}$$

i. **Is $\vec{v}_3$ in span$\{\vec{v}_1, \vec{v}_2\}$? Justify your answer.**

ii. **Is the set of vectors,$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$? If not is it a basis for another subspace?**

Solution:

i. To check if $\begin{bmatrix} 4.5 \\ 6 \\ 1 \end{bmatrix}$ is in span$\left\{\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}\right\}$, we have to check if $\begin{bmatrix} 4.5 \\ 6 \\ 1 \end{bmatrix}$ is a linear combination of the vectors in the span. i.e. Does $\begin{bmatrix} 4.5 \\ 6 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}$ have at least one solution for α and β? We can solve using Gaussian Elimination.

$$\left[\begin{array}{cc|c} 1 & 3 & 4.5 \\ 4 & 6 & 6 \\ 6 & 5 & 1 \end{array}\right] \xrightarrow{R_2=-\frac{1}{6}(R_2-4R_1)} \left[\begin{array}{cc|c} 1 & 3 & 4.5 \\ 0 & 1 & 2 \\ 6 & 5 & 1 \end{array}\right] \xrightarrow{R_3=R_3-6R_1} \left[\begin{array}{cc|c} 1 & 3 & 4.5 \\ 0 & 1 & 2 \\ 0 & -13 & -26 \end{array}\right] \xrightarrow{R_3=R_3+13R_2}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 4.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1=R_1-3R_2} \left[\begin{array}{cc|c} 1 & 0 & -1.5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Since we found a unique solution for $\alpha = -1.5$ and $\beta = 2$, we can conclude that $\begin{bmatrix} 4.5 \\ 6 \\ 1 \end{bmatrix}$ is in the span.

ii. From the previous part, we saw that $\vec{v}_3 \in$ span$\{\vec{v}_1, \vec{v}_2\}$ therefore the three vectors are not linearly independent so they cannot be a basis for $\mathbb{R}^3$ or for any other vector space.

(d) (4 points) **Calculate the nullspace of the matrix:**

$$\begin{bmatrix} 1 & -1 & 4 & -8 \\ 0 & 1 & -2 & 5 \end{bmatrix}$$

Solution: To find the nullspace we find all vectors that will be mapped to the null vector, so we can set up the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & -1 & 4 & -8 & 0 \\ 0 & 1 & -2 & 5 & 0 \end{array}\right]$$

Then, we use Gaussian elimination to reduce the matrix to reduced row echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & 4 & -8 & 0 \\ 0 & 1 & -2 & 5 & 0 \end{array}\right] \xrightarrow{R_1=R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & -2 & 5 & 0 \end{array}\right]$$

Now, since the variable 3 and 4 have no pivots, they are the free variables, and we let $x_3 = s, s \in \mathbb{R}$ and $x_4 = t, t \in \mathbb{R}$. We can now express $\vec{x}$ in terms of s and t:

$$\begin{cases} x_1 = -2s + 3t \\ x_2 = 2s - 5t \\ x_3 = s \\ x_4 = t \end{cases},$$

so

$$\vec{x} = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} t.$$

Therefore, the nullspace of this matrix is:

$$\text{span}\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(e) (6 points) For the following bases given by **U** and **V**

$$\mathbf{U} = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

the vectors $\vec{r}^{(u)}$ and $\vec{r}^{(v)}$ are defined as the vector $\vec{r}$ coordinated in the $\mathbb{U}$ and $\mathbb{V}$ bases respectively. In other words

$$\vec{r} = r_1^{(u)}\vec{u}_1 + r_2^{(u)}\vec{u}_2 + \ldots + r_n^{(u)}\vec{u}_n = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} r_1^{(u)} \\ r_2^{(u)} \\ \vdots \\ r_n^{(u)} \end{bmatrix} = \mathbf{U}\vec{r}^{(u)}$$

$$= r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \ldots + r_n^{(v)}\vec{v}_n = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \\ \vdots \\ r_n^{(v)} \end{bmatrix} = \mathbf{V}\vec{r}^{(v)}$$

i. **Calculate the coordinate transformation between the $\mathbb{U}$ and $\mathbb{V}$ bases.**
i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$.

ii. **If $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute $\vec{r}^{(v)}$.**

Solution:

i.

$$\mathbf{T} = \mathbf{V}^{-1}\mathbf{U} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 4 & 2 \end{bmatrix}$$

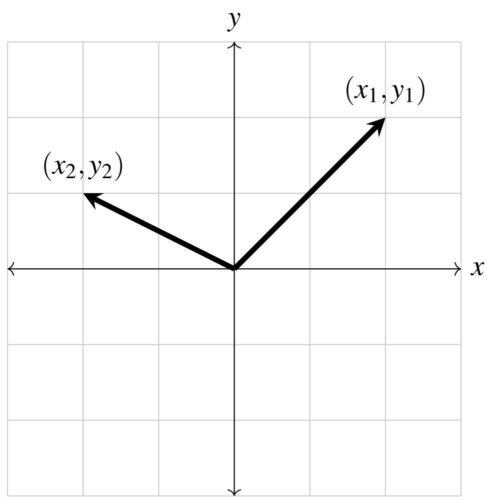
ii.

$$\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)} = \begin{bmatrix} -3 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

4. Swimming Synchronously (15 points) You are the choreographer for a synchronized swimming team! To model the choreography, you have decided to represent the swimmers' locations as vectors in $\mathbb{R}^2$, where the first entry represents the x-coordinate of the swimmer's position, and the second entry represents the y-coordinate. $(0,0)$ represents the center of the pool. For example, the matrix

$$\mathbf{M} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

would represent swimmers in the following two locations:



(a) (3 points) Let $\mathbf{M}$ be a matrix in $\mathbb{R}^{2\times 2}$ containing the positions of two swimmers as its columns. You would like the first swimmer to swim 2 units in the positive x-direction and 3 units in the negative y-direction, and the second swimmer to swim 4 units in the positive x-direction and 1 unit in the positive y-direction. Let $\mathbf{F}$ be the matrix containing the positions of the swimmers after this transformation. **Write an expression for F in terms of M.** *Hint: Consider which operation will change each coordinate by a constant amount, no matter where the swimmer originally starts.*

Solution: The transformation of shifting a vector is best described by vector addition, so in order to perform a vector addition to both columns of $\mathbf{M}$ we will add a matrix to it containing the appropriate shift for each element of $\mathbf{M}$. We can then express $\mathbf{F}$ as follows:

$$\mathbf{F} = \mathbf{M} + \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

(b) (3 points) Let Routine A be a routine described by a matrix $\mathbf{A}$ such that a swimmer who performs the routine beginning at initial position $\vec{p}$ will end up at position $\mathbf{A}\vec{p}$, and let Routine B be a routine described by the matrix $\mathbf{B}$ such that a swimmer at initial position $\vec{q}$ will end up at the position $\mathbf{B}\vec{q}$. Let

M again be a matrix containing the initial positions of swimmers as its columns. **Write an expression for G, the matrix containing the final positions of each swimmer after performing first Routine A and then Routine B.** Your answer should be in terms of **A**, **B**, and **M**.

Solution: To first model the effect of performing Routine A, we can express the result of multiplying each swimmer's position by **A** as **AM**. Similarly, we then multiply by **B** to get the result after performing both routines, so $\mathbf{G} = \mathbf{BAM}$.

(c) (4 points) You are teaching the swimmers a new move where they all rotate 60°counter clockwise around the center of the pool, and then swim half of the way to the center of the pool from their current location. **Create a matrix C such that a swimmer who performs this move starting from arbitrary location $\vec{p}$ ends up at the location represented by $\mathbf{C}\vec{p}$.** For reference, the general form of a rotation matrix is given below. *Note: You do not need to compute sines and cosines.*

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Solution: First, we can express the rotation by 60°by substituting for θ in the **R** matrix, to get the matrix $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Next we can model swimming halfway to the center of the pool as scaling the swimmers positions by half, which can be expressed by the matrix $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. Finally, we can compose these two operations into a single matrix.

$$\mathbf{C} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

(d) (5 points) A colleague from the Stanford synchronized swimming team recommends you use a routine represented by the following matrix **D**:

$$\mathbf{D} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

However, your TA warns you that this routine may cause swimmers to crash into each other by mapping swimmers from different initial positions to the same final positions. You want to find out which sets of points will map to the same location. Specifically, **for a given starting point $\vec{x}$, find the set of all initial positions $\vec{v}$ that map to the same final position as $\vec{x}$,** i.e. $\mathbf{D}\vec{x} = \mathbf{D}\vec{v}$.

Solution: In order for $\vec{x}$ and $\vec{v}$ to map to the same output value, we need $\mathbf{D}\vec{x} - \mathbf{D}\vec{v} = \mathbf{D}(\vec{x} - \vec{v}) = \vec{0}$. Therefore, we need $\vec{x} - \vec{v}$ to belong to the nullspace of **D**. By inspection or by row-reduction, we can find that the nullspace of **D** is span $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$. Therefore, adding any scalar multiple of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ to $\vec{x}$ won't change the final position where the vector is mapped. Therefore, we can express the set of all possible values for $\vec{v}$ as $\vec{x} + \alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, where $\alpha \in \mathbb{R}$.

5. **Ground Control to Prof. Waller (16 points)**

One of the many challenges in robotics is building efficient and lightweight systems. We want to build systems that accomplish all our goals while keeping the cost (and often number of parts) low.

The latest and greatest startup SpaceWhy wants to build low cost spaceships. Specifically, given some prescribed targets or destinations, they want to figure out the minimum number of rocket boosters needed to get to that target. They hire you with your EECS16A skills to help pick the booster orientations to accomplish this goal.

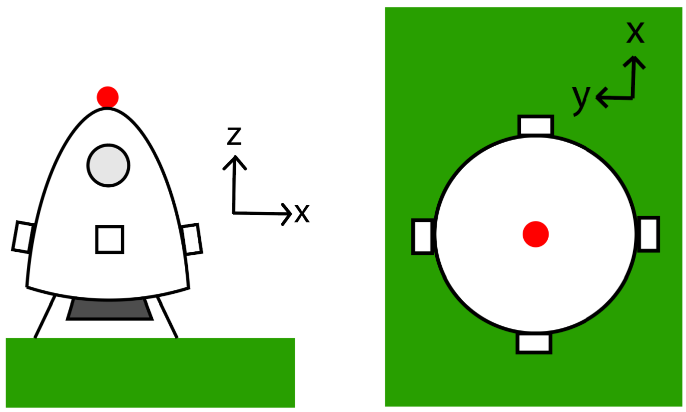


Figure 5.1: SpaceWhy's spaceship, side and top views. Bottom and side boosters shown

(a) (2 points) With just one rocket booster $\vec{b}_1$ on the bottom of the ship, we can make a controlled launch in the z-direction represented by $\vec{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. **Define the vector space of the directions in which we can travel. What is its dimension?**

Hint: Mathematically express the set of vector directions that we can travel to.

Solution: The vector space can be described as span$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Its dimension is 1.

(b) (4 points) We want to reach the moon, which relative to our launch pad is at location $\vec{m} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Using an additional booster mounted to the side of the space ship, we are able to travel in the direction described by $\vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. **With boosters $\vec{b}_1$ and $\vec{b}_2$ are we able to reach the moon? If so, find the coefficients needed to reach the moon. If not, explain why.**

Solution: Yes because the moon location is a linear combination of the direction vectors of our boosters. i.e. $\vec{m} = \alpha\vec{b}_1 + \beta\vec{b}_2$ Using Gaussian Elimination we can solve for α and β.

$$\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{swap } R_1, R_3} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

We can see that $\alpha = 2$ and $\beta = 1$ so our target location is indeed a linear combination of the direction vectors of the boosters.

(c) (3 points) **With boosters in the $\vec{b}_1$ and $\vec{b}_2$ directions, are we able to travel anywhere we want in space? Why or why not?**

Solution: No. We cannot travel anywhere we want in space because we need at least 3 linearly independent vectors to span $\mathbb{R}^3$

(d) (4 points) We have now set up the rocket boosters to aim in 3 directions represented by vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \vec{b}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Can we reach the point given by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ **? Justify your answer.**

Solution: To show that we can reach the point, it is sufficient to find $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 \vec{b}_1 + \alpha_2 \vec{b}_2 + \alpha_3 \vec{b}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. We can solve this using Gaussian Elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_3 = \frac{1}{2} R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 = \frac{1}{2} R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We can reach $\begin{bmatrix}1\\0\\2\end{bmatrix}$ using $\frac{1}{2}\vec{b}_1+\frac{1}{2}\vec{b}_2+\vec{b}_3$.

(e) (3 points) Your friend suggests adding a 4th rocket booster $\vec{b}_4=\begin{bmatrix}1\\-2\\0\end{bmatrix}$ in addition to your existing boosters from the previous part. **Will this allow you to reach more locations? Why or why not?**

Solution: No, the three vectors from the previous part already span $\mathbb{R}^3$, adding another vector will make the set linearly dependent and will not allow us to cover more locations.

6. **Steady the Traffic (29 points)**

(a) (4 points) Your friend wants to study the flow of traffic around the Bay Area and asks for your help. From her observations, your friend finds that the number of cars in San Francisco, Berkeley, San Jose and Fremont can be represented in the following way:

$$\begin{bmatrix} x_{\text{SF}}[n+1] \\ x_{\text{B}}[n+1] \\ x_{\text{SJ}}[n+1] \\ x_{\text{F}}[n+1] \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{\text{SF}}[n] \\ x_{\text{B}}[n] \\ x_{\text{SJ}}[n] \\ x_{\text{F}}[n] \end{bmatrix} \tag{1}$$

The flow of traffic is represented in the diagram below. **Write the transition matrix** A **corresponding to this diagram.**

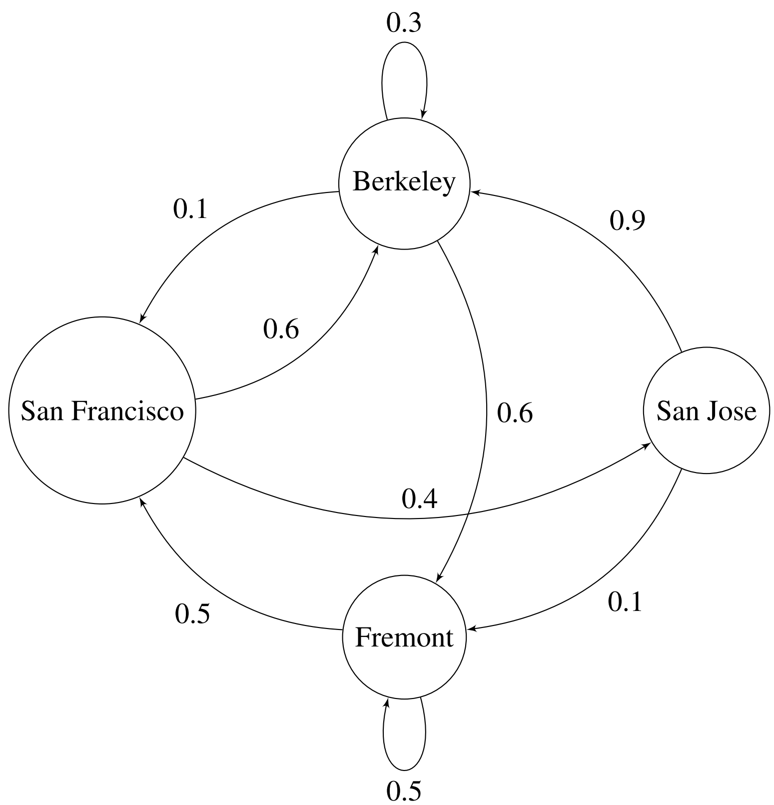


Figure 6.1: A flow diagram to represent how model **A** transforms state vector $\vec{x}[n]$.

Solution: Using the flow diagram, we can first write out the state equations as follows:

$$\begin{aligned} x_{SF}[n+1] &= 0.1x_B[n] + 0.5x_F[n] \\ x_B[n+1] &= 0.6x_{SF}[n] + 0.3x_B[n] + 0.9x_{SJ}[n] \\ x_{SJ}[n+1] &= 0.4x_{SF}[n] \\ x_F[n+1] &= 0.6x_B[n] + 0.1x_{SJ}[n] + 0.5x_F[n] \end{aligned}$$

Filling in the elements of $\mathbf{A}$ accordingly gives the following result for the transition matrix of the system above:

$$\mathbf{A} = \begin{bmatrix} 0 & 0.1 & 0 & 0.5 \\ 0.6 & 0.3 & 0.9 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0.1 & 0.5 \end{bmatrix}.$$

(b) (4 points) Your friend takes measurements of the number of cars at the first 3 cities (San Francisco, Berkeley, San Jose) during Thanksgiving weekend and finds the following transition matrix:

$$\mathbf{T} = \begin{bmatrix} 0.25 & 0 & 0.3 \\ 0.75 & 1 & 0.4 \\ 0 & 0 & 0.3 \end{bmatrix}$$

The new state vector is:

$$\vec{x}[n] = \begin{bmatrix} x_{\text{SF}}[n] \\ x_{\text{B}}[n] \\ x_{\text{SJ}}[n] \end{bmatrix}.$$

You are performing some simulations to see how the traffic evolves at each time step. You start your simulation with 200 cars at San Francisco, 150 cars at Berkeley and 100 cars at San Jose. **Calculate the number of cars at each city in the next time step.**

Solution: We need to apply $\mathbf{T}$ to $\vec{x}[0]$ to calculate the next state vector:

$$\vec{x}[1] = \mathbf{T} \cdot \vec{x}[0] = \begin{bmatrix} 0.25 & 0 & 0.3 \\ 0.75 & 1 & 0.4 \\ 0 & 0 & 0.3 \end{bmatrix} \begin{bmatrix} 200 \\ 150 \\ 100 \end{bmatrix} = \begin{bmatrix} 80 \\ 340 \\ 30 \end{bmatrix}$$

(c) (5 points) It would be helpful for your simulations to know the eigenvectors of this transition matrix. **Calculate the steady state eigenvector associated with the eigenvalue λ = 1 for the above matrix T.**

Solution:

$$\lambda = 1: \begin{bmatrix} 0.25 & 0 & 0.3 \\ 0.75 & 1 & 0.4 \\ 0 & 0 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -0.75 & 0 & 0.3 \\ 0.75 & 0 & 0.4 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies$$

$$\begin{bmatrix} -0.75 & 0 & 0 \\ 0.75 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \text{x = 0, z = 0 and y is a free variable}$$

Any vector that lies in span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is an eigenvector corresponding to eigenvalue $\lambda = 1$.

(d) (6 points) Next you are interested in investigating the traffic flows during New Year's weekend. Your friend tells you the following information about the transition matrix for this period:

i. The transition matrix is conservative

ii. The eigenvector corresponding to $\lambda = 1$ is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

iii. All other eigenvalues $|\lambda_i| < 1$

If the initial state vector is $\vec{x}[0] = \begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix}$, **what steady state will this system converge to?**

Solution: Since the transition matrix is conservative, we know that the total amount of cars driving will not change. From our initial state vector we can determine that the total amount of cars is $30+50+20 = 100$. Additionally, because we know that our steady state vector for $\lambda = 1$ is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, we know the steady state of this system will be in span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Because the total amount of cars is always 100, we can conclude that our steady state vector is $\begin{bmatrix} 50 \\ 0 \\ 50 \end{bmatrix}$.

(e) (10 points) Next, for a new transition matrix **S**, you investigate the traffic flow of commuters throughout the year. You calculate the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 0.25$ with corresponding eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For the given values of $\vec{x}[0]$, write down whether the system will converge to a non-zero steady state, decay to zero or keep growing infinitely.

i. If $\vec{x}[0] = \begin{bmatrix} 350 \\ 50 \\ 300 \end{bmatrix}$

ii. If $\vec{x}[0] = \begin{bmatrix} 15 \\ 10 \\ 25 \end{bmatrix}$

Solution: Because our system has 3 linearly independent eigenvectors, any initial state vector $\vec{x}[0]$ can be written as a linear combination of them. i.e. $x[0] = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3$

To determine whether or not a system will converge over time, we must look at $\lim_{n\to\infty} \vec{x}[n]$ where $\vec{x}[n] = S^n \vec{x}[0]$.We can then calculate:

$$\begin{aligned} S^n \vec{x}[0] &= S^n(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3) \\ &= \alpha_1 S^n \vec{v}_1 + \alpha_2 S^n \vec{v}_2 + \alpha_3 S^n \vec{v}_3 \\ &= (1)^n \alpha_1 \vec{v}_1 + (4)^n \alpha_2 \vec{v}_2 + (0.25)^n \alpha_3 \vec{v}_3 \end{aligned}$$

i. For $\vec{x}[0] = \begin{bmatrix} 350 \\ 50 \\ 300 \end{bmatrix}$, we can see that $\vec{x}[0] = 300\vec{v}_1 + 0\vec{v}_2 + 50\vec{v}_3$ As a result

$$\lim_{n\to\infty} \vec{x}[n] = \lim_{n\to\infty}((1)^n 300\vec{v}_1 + (0.25)^n 50\vec{v}_3)$$

will converge to a non-zero steady state.

ii. For $\vec{x}[0] = \begin{bmatrix} 15 \\ 10 \\ 25 \end{bmatrix}$, we can see that $\vec{x}[0] = 15\vec{v}_1 + 10\vec{v}_2 + 0\vec{v}_3$ As a result

$$\lim_{n\to\infty} \vec{x}[n] = \lim_{n\to\infty}((1)^n 15\vec{v}_1 + (4)^n 10\vec{v}_2)$$

which will continue to grow infinitely.

7. A Problem N(o)-body Can Solve (24 points)

An N-body simulation is a method of modeling the interactions between a set of particles, and it is commonly implemented in an astrophysics context to study the movements of celestial bodies and galactic formation under the constraints of gravitational forces. In each timestep, the core algorithm iterates through particle pairs to calculate the force on each particle and update its current position.

As part of your work in a research lab, you are developing an efficient N-body simulation for the solar system that exploits computationally fast operations on matrices to speed up runtime (good thing you're taking EECS16A!). You represent each body as a vector in 3D space:

$$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

You calculate that the position of one particular body – Earth – is updated in the following way during every timestep:

- $x[t+1] = 0.5x[t] + 0.7y[t] + 0.3z[t]$
- $y[t+1] = 0.6y[t] + 0.1z[t]$
- $z[t+1] = 0.3x[t] + 0.2y[t] + z[t]$

(a) (4 points) After one timestep, at time $t = 1$, Earth is located at $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. You want to calculate the position of Earth at $t = 0$. **Formulate this problem as a matrix-vector equation in the form $A\vec{x} = \vec{b}$.** You do not need to solve for Earth's position.

Solution: The system of equations representing the position of Earth between timesteps t and $t+1$ can be substituted with $t = 0$ and $t+1 = 1$ respectively. We can then rewrite the system so $\vec{x} = \vec{p}_{\text{Earth}}[0]$ (the unknown initial position of Earth), $\vec{b} = \vec{p}_{\text{Earth}}[1]$ (the known position of Earth at $t = 1$), and A represents the coefficients relating $\vec{p}_{\text{Earth}}[1]$ to $\vec{p}_{\text{Earth}}[0]$:

$$\mathbf{A}\vec{p}_{\text{Earth}}[0] = \vec{p}_{\text{Earth}}[1]$$

$$\begin{bmatrix} 0.5 & 0.7 & 0.3 \\ 0 & 0.6 & 0.1 \\ 0.3 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix} = \begin{bmatrix} x[1] \\ y[1] \\ z[1] \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0.7 & 0.3 \\ 0 & 0.6 & 0.1 \\ 0.3 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},$$

substituting for the Earth's known position at $t = 1$.

(b) (6 points) You have determined that Neptune's position in each timestep is updated according to the following matrix:

$$\mathbf{N} = \begin{bmatrix} 1 & 0.2 & 0 \\ 0 & -0.2 & 0.1 \\ -1 & 0 & 0.1 \end{bmatrix}$$

You let the simulation run in the background for a while, and at $t = n$, Neptune is located at $\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$. You then realize that you've forgotten to record position data since you started! **Is it possible to recover Neptune's position uniquely at $t = n-1$?** If it is, **use Gaussian elimination to find the inverse of $\mathbf{N}$, $\mathbf{N}^{-1}$.**

Solution: We can use Gaussian elimination to calculate N^{-1} by augmenting the N matrix with the identity matrix, and row-reducing N until it is transformed into the identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ -1 & 0 & 0.1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3=R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0.2 & 0.1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{R_3=R_3+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0 & 0.2 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3=5R_3} \left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array}\right] \xrightarrow{R_1=R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array}\right] \xrightarrow{R_2=10R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 10 & 0 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array}\right] \xrightarrow{R_2=R_2-R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -5 & 5 & -5 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array}\right] \xrightarrow{R_2=-\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2.5 & -2.5 & 2.5 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array}\right] \xrightarrow{R_1=R_1-\frac{1}{10}R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 1 & 0 & 2.5 & -2.5 & 2.5 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array}\right]$$

The $\mathbf{N}^{-1}$ matrix we can use to recover Neptune's previous positions is:

$$\mathbf{N}^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 2.5 & -2.5 & 2.5 \\ 5 & 5 & 5 \end{bmatrix}$$

(c) (8 points) Additionally, you have the following matrix for Pluto:

$$\begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

If Pluto is positioned at some unspecified $\begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix}$ at $t = 0$, **are there any points in $\mathbb{R}^3$ space that you cannot reach at at $t = 1$? If so, what is the subspace that Pluto can be located in?**

Note: You do not have to provide rigorous justification.

Solution: Performing Gaussian elimination on Pluto's update matrix reveals that the rows (and columns) are linearly dependent, indicating that the columns cannot span $\mathbb{R}^3$.

$$\begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because one row was reduced to 0's during the process of row-reduction, the subspace Pluto's position can be in at $t = 1$ is two-dimensional. We can select any two column vectors from the matrix to represent the span of this subspace; a possible solution is:

$$\text{span}\left\{ \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix} \right\}$$

(d) (6 points) After running your simulation repeatedly, you notice that with the current update matrices you have entered, Venus and Mars are all moving within the same 2D orbital plane. You refer back to your calculations, but you notice there is a smudge obscuring one element of matrix $\mathbf{M}$:

$$\mathbf{V} = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0 & 0.7 & 0.7 \\ 0.7 & 1.1 & 0.4 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0.4 & 0.6 & 0.2 \\ -1.4 & 0 & 1.4 \\ 0.6 & m_{32} & 0.8 \end{bmatrix}$$

$\mathbf{V}$ and $\mathbf{M}$ are the update matrices for Venus and Mars respectively. **Fill in the missing matrix element (denoted by "m_{32}") in a way that would explain the behavior of these 2 planets.**

Solution: $m_{32} = 1.4$ For both planets to be moving on the same 2D plane within $\mathbb{R}^3$, all three update matrices must contain exactly two linearly independent column vectors that span the plane. These spans must be equivalent between the independent column vectors for V and M.

We denote C_i as the ith column of a mentioned matrix. By inspection, we see that $C_3 = C_2 - C_1$ for V, $C_1 = C_2 - C_3$ for M. This implies that $? = 0.8 + 0.6 = 1.4$ in M. Replacing the ? with this value would create a scenario in which Venus and Mars can reach any position in the 2D plane defined by the span below:

$$\text{span}\left\{ \begin{bmatrix} 0.3 \\ 0 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0.4 \\ 0.7 \\ 1.1 \end{bmatrix} \right\}$$

8. Proof (10 points)

You are told that a $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a conservative transition matrix. **Prove that it has an eigenvalue of λ = 1.**

Solution: First, we must start with what we know, which is that $A \in \mathbb{R}^{2\times 2}$ is a conservative transition matrix.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a + c = 1$ and $b + d = 1$.

Here are two possible ways you can prove this:

i The more challenging way is to solve directly for the eigenvalues of A using the $\det(A - \lambda I) = 0$ formula and substituting in $b = 1 - d$ and $c = 1 - a$ where appropriate.

$$\begin{aligned} \det(A - \lambda I) &= (a - \lambda)(d - \lambda) - bc \\ &= ad - (a + d)\lambda + \lambda^2 - (1 - a)(1 - d) \\ &= ad - (a + d)\lambda + \lambda^2 - (1 - a - d + ad) \\ &= \lambda^2 - (a + d)\lambda + (-1 + a + d) \\ &= (\lambda - 1)(\lambda - (-1 + a + d)) \end{aligned}$$

From here we can see that $\lambda = 1$ is a root of this characteristic equation.

ii A quicker method that we may notice is that we are only looking to show that $\lambda = 1$ is an eigenvalue of A, not find all eigenvalues. If we assume that A does have eigenvalue of 1, then we know that $\det(A - I) = 0$. Using our knowledge that $b = 1 - d$ and $c = 1 - a$ we can show this is true.

$$\begin{aligned} \det(A - I) &= (a - 1)(d - 1) - bc \\ &= (a - 1)(d - 1) - (1 - a)(1 - d) \\ &= 0 \end{aligned}$$