EECS 16A Designing Information Devices and Systems I
Spring 2021 Midterm 1 Instructions

Goodluck for the midterm! You've studied hard and we are rooting for you to do well! Please read these instructions and the proctoring guidelines before the exam.

Our advice to you: if you can't solve a particular problem, move on to another, or state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution. **Good luck! We believe in you.**

Format & How to Submit Answers

There are 8 problems (2 introductory questions, and 6 exam questions with subparts) of varying numbers of points on the exam. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can. If you are having trouble with one problem, there may be easier points available later in the exam!

Complete your exam using either the template provided or appropriately created sheets of paper. Either way, you should submit your answers to the Gradescope assignment that is marked Midterm 1 for your specific exam group. Make sure you submit your assignment to the correct Gradescope assignment. You MUST select pages for each question. We cannot grade your exam if you do not select pages for each question. If you are having technical difficulties submitting your exam, you can email your answers to eecs16a@berkeley.edu.

In general, show all your work legibly to receive full credit; we cannot grade anything that we cannot read. For some problems, we may try to award partial credit for substantial progress on a problem, and showing your work clearly and legibly will help us do that.

Timing & Academic Honesty

You are expected to follow the rules provided in the Exam Proctoring Guidelines (`https://docs.google.com/document/d/10pnWwxyZ40nlpbCM4aOYTxXOjc36sIQaMx9m8zyaR8w/edit?usp=sharing`). The exam will be available to you at the link sent to you by email. The exam will start at 7pm Pacific Time, Monday, March 1st 2021, unless you have an exam accommodation. If you experience technical difficulties and cannot access your exam, let us know by making a private post on Piazza and we will try to help.

You have 120 minutes for the exam, with 40 extra minutes for scanning and submitting to Gradescope. Unless you have an accommodation, you must submit your exam by 9:40pm. Late submissions will be penalized exponentially. An exam that is submitted N minutes after the end of the submission period will lose 2^N points. This means that if you are 1 minute late you will lose 2 points; if you are 5 minutes late you will lose 32 points and so on.

This is a closed-note, closed-book, closed-internet, and closed-collaboration exam. Calculators are not allowed. You may consult a single handwritten 8.5" by 11" cheat sheet (front and back). Do not attempt to cheat in any way. We have a zero tolerance policy for violations of the Berkeley Honor Code.

EECS 16A Designing Information Devices and Systems I
Spring 2021 Midterm 1

1. HONOR CODE

If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my one reference cheat sheet.*
- *I did not collaborate with any other human being on this exam.*

2. (a) **Tell us about something that makes you happy.** *All answers will be awarded full credit; you can be brief.* **(2 Points)**

(b) **What is one of your hobbies?** *All answers will be awarded full credit; you can be brief.* **(2 Points)**

3. Mechanical Linear Algebra (22 points)

(a) (4 points) Consider the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 - x_2 + x_3 = 2 \\ -2x_1 + 2x_2 - 2x_3 = -4 \\ 2x_1 + 2x_3 = 3 \end{cases}$$

Write the system of equations in augmented matrix form and bring to **reduced row echelon form** through Gaussian elimination. **How many solutions (if any) does this system of equations have?**

(b) (2 points) For the new row reduced matrix below:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & d-5 \end{array}\right]$$

For what value of d (if any) will the system have a solution? If it does have a solution, what is the solution?

(c) (6 points) For vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ shown below:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ 6 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 4.5 \\ 6 \\ 1 \end{bmatrix}$$

i. **Is $\vec{v}_3$ in span$\{\vec{v}_1, \vec{v}_2\}$? Justify your answer.**

ii. **Is the set of vectors,$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a basis for $\mathbb{R}^3$? If not is it a basis for another subspace?**

(d) (4 points) **Calculate the nullspace of the matrix:**

$$\begin{bmatrix} 1 & -1 & 4 & -8 \\ 0 & 1 & -2 & 5 \end{bmatrix}$$

(e) (6 points) For the following bases given by $\mathbf{U}$ and $\mathbf{V}$

$$\mathbf{U} = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$

the vectors $\vec{r}^{(u)}$ and $\vec{r}^{(v)}$ are defined as the vector $\vec{r}$ coordinated in the $\mathbb{U}$ and $\mathbb{V}$ bases respectively. In other words

$$\vec{r} = r_1^{(u)}\vec{u}_1 + r_2^{(u)}\vec{u}_2 + \ldots + r_n^{(u)}\vec{u}_n = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} r_1^{(u)} \\ r_2^{(u)} \\ \vdots \\ r_n^{(u)} \end{bmatrix} = \mathbf{U}\vec{r}^{(u)}$$

$$= r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \ldots + r_n^{(v)}\vec{v}_n = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} r_1^{(v)} \\ r_2^{(v)} \\ \vdots \\ r_n^{(v)} \end{bmatrix} = \mathbf{V}\vec{r}^{(v)}$$

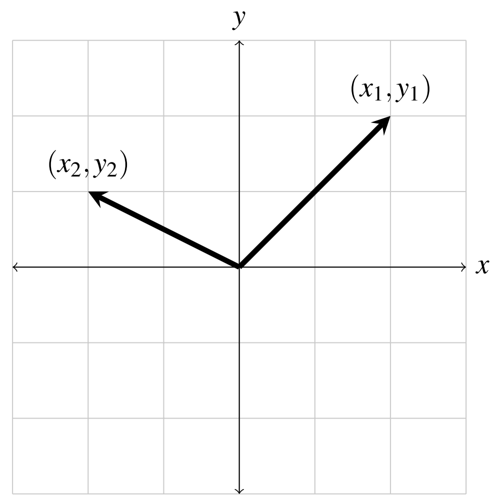
i. **Calculate the coordinate transformation between the $\mathbb{U}$ and $\mathbb{V}$ bases.**
   i.e. find a matrix $\mathbf{T}$, such that $\vec{r}^{(v)} = \mathbf{T}\vec{r}^{(u)}$.

ii. **If $\vec{r}^{(u)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, compute $\vec{r}^{(v)}$.**

4. Swimming Synchronously (15 points) You are the choreographer for a synchronized swimming team! To model the choreography, you have decided to represent the swimmers' locations as vectors in $\mathbb{R}^2$, where the first entry represents the x-coordinate of the swimmer's position, and the second entry represents the y-coordinate. $(0,0)$ represents the center of the pool. For example, the matrix

$$\mathbf{M} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

would represent swimmers in the following two locations:



(a) (3 points) Let $\mathbf{M}$ be a matrix in $\mathbb{R}^{2\times 2}$ containing the positions of two swimmers as its columns. You would like the first swimmer to swim 2 units in the positive x-direction and 3 units in the negative y-direction, and the second swimmer to swim 4 units in the positive x-direction and 1 unit in the positive y-direction. Let $\mathbf{F}$ be the matrix containing the positions of the swimmers after this transformation. **Write an expression for F in terms of M.** *Hint: Consider which operation will change each coordinate by a constant amount, no matter where the swimmer originally starts.*

(b) (3 points) Let Routine A be a routine described by a matrix $\mathbf{A}$ such that a swimmer who performs the routine beginning at initial position $\vec{p}$ will end up at position $\mathbf{A}\vec{p}$, and let Routine B be a routine described by the matrix $\mathbf{B}$ such that a swimmer at initial position $\vec{q}$ will end up at the position $\mathbf{B}\vec{q}$. Let $\mathbf{M}$ again be a matrix containing the initial positions of swimmers as its columns. **Write an expression for G, the matrix containing the final positions of each swimmer after performing first Routine A and then Routine B.** Your answer should be in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{M}$.

(c) (4 points) You are teaching the swimmers a new move where they all rotate 60°counter clockwise around the center of the pool, and then swim half of the way to the center of the pool from their current location. **Create a matrix C such that a swimmer who performs this move starting from arbitrary location $\vec{p}$ ends up at the location represented by $\mathbf{C}\vec{p}$.** For reference, the general form of a rotation matrix is given below. *Note: You do not need to compute sines and cosines.*

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(d) (5 points) A colleague from the Stanford synchronized swimming team recommends you use a routine represented by the following matrix $\mathbf{D}$:

$$\mathbf{D} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

However, your TA warns you that this routine may cause swimmers to crash into each other by mapping swimmers from different initial positions to the same final positions. You want to find out which sets of points will map to the same location. Specifically, **for a given starting point $\vec{x}$, find the set of all initial positions $\vec{v}$ that map to the same final position as $\vec{x}$,** i.e. $\mathbf{D}\vec{x} = \mathbf{D}\vec{v}$.

5. Ground Control to Prof. Waller (16 points)

One of the many challenges in robotics is building efficient and lightweight systems. We want to build systems that accomplish all our goals while keeping the cost (and often number of parts) low.

The latest and greatest startup SpaceWhy wants to build low cost spaceships. Specifically, given some prescribed targets or destinations, they want to figure out the minimum number of rocket boosters needed to get to that target. They hire you with your EECS16A skills to help pick the booster orientations to accomplish this goal.

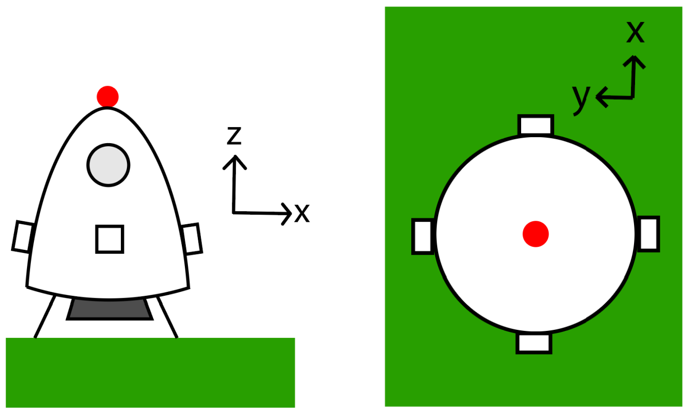


Figure 5.1: SpaceWhy's spaceship, side and top views. Bottom and side boosters shown

(a) (2 points) With just one rocket booster $\vec{b}_1$ on the bottom of the ship, we can make a controlled launch in the z-direction represented by $\vec{b}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. **Define the vector space of the directions in which we can travel. What is its dimension?**

Hint: Mathematically express the set of vector directions that we can travel to.

(b) (4 points) We want to reach the moon, which relative to our launch pad is at location $\vec{m} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Using an additional booster mounted to the side of the space ship, we are able to travel in the direction described by $\vec{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. **With boosters $\vec{b}_1$ and $\vec{b}_2$ are we able to reach the moon? If so, find the coefficients needed to reach the moon. If not, explain why.**

(c) (3 points) **With boosters in the $\vec{b}_1$ and $\vec{b}_2$ directions, are we able to travel anywhere we want in space? Why or why not?**

(d) (4 points) We have now set up the rocket boosters to aim in 3 directions represented by vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \vec{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \vec{b}_3 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Can we reach the point given by $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$**? Justify your answer.**

(e) (3 points) Your friend suggests adding a 4th rocket booster $\vec{b}_4 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ in addition to your existing boosters from the previous part. **Will this allow you to reach more locations? Why or why not?**

6. Steady the Traffic (29 points)

(a) (4 points) Your friend wants to study the flow of traffic around the Bay Area and asks for your help. From her observations, your friend finds that the number of cars in San Francisco, Berkeley, San Jose and Fremont can be represented in the following way:

$$\begin{bmatrix} x_{\mathrm{SF}}[n+1] \\ x_{\mathrm{B}}[n+1] \\ x_{\mathrm{SJ}}[n+1] \\ x_{\mathrm{F}}[n+1] \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{\mathrm{SF}}[n] \\ x_{\mathrm{B}}[n] \\ x_{\mathrm{SJ}}[n] \\ x_{\mathrm{F}}[n] \end{bmatrix} \tag{1}$$

The flow of traffic is represented in the diagram below. **Write the transition matrix** A **corresponding to this diagram.**

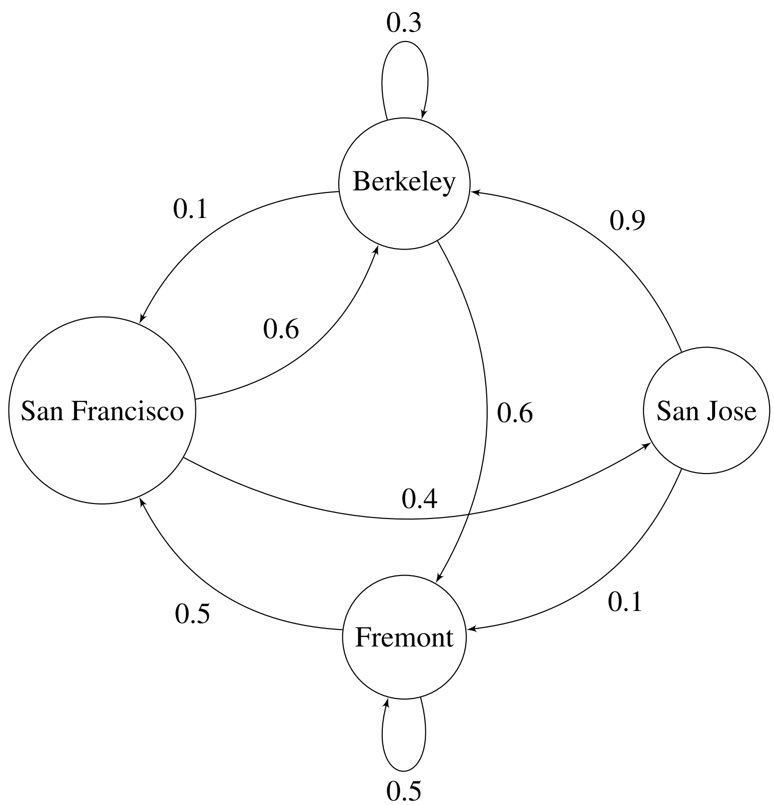


Figure 6.1: A flow diagram to represent how model **A** transforms state vector $\vec{x}[n]$.

(b) (4 points) Your friend takes measurements of the number of cars at the first 3 cities (San Francisco, Berkeley, San Jose) during Thanksgiving weekend and finds the following transition matrix:

$$\mathbf{T} = \begin{bmatrix} 0.25 & 0 & 0.3 \\ 0.75 & 1 & 0.4 \\ 0 & 0 & 0.3 \end{bmatrix}$$

The new state vector is:

$$\vec{x}[n] = \begin{bmatrix} x_{\mathrm{SF}}[n] \\ x_{\mathrm{B}}[n] \\ x_{\mathrm{SJ}}[n] \end{bmatrix}.$$

You are performing some simulations to see how the traffic evolves at each time step. You start your simulation with 200 cars at San Francisco, 150 cars at Berkeley and 100 cars at San Jose. **Calculate the number of cars at each city in the next time step.**

(c) (5 points) It would be helpful for your simulations to know the eigenvectors of this transition matrix. **Calculate the steady state eigenvector associated with the eigenvalue λ = 1 for the above matrix T.**

(d) (6 points) Next you are interested in investigating the traffic flows during New Year's weekend. Your friend tells you the following information about the transition matrix for this period:

i. The transition matrix is conservative

ii. The eigenvector corresponding to $\lambda = 1$ is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

iii. All other eigenvalues $|\lambda_i| < 1$

If the initial state vector is $\vec{x}[0] = \begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix}$, **what steady state will this system converge to?**

(e) (10 points) Next, for a new transition matrix **S**, you investigate the traffic flow of commuters throughout the year. You calculate the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 0.25$ with corresponding eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For the given values of $\vec{x}[0]$, write down whether the system will converge to a non-zero steady state, decay to zero or keep growing infinitely.

i. If $\vec{x}[0] = \begin{bmatrix} 350 \\ 50 \\ 300 \end{bmatrix}$

ii. If $\vec{x}[0] = \begin{bmatrix} 15 \\ 10 \\ 25 \end{bmatrix}$

7. A Problem N(o)-body Can Solve (24 points)

An N-body simulation is a method of modeling the interactions between a set of particles, and it is commonly implemented in an astrophysics context to study the movements of celestial bodies and galactic formation under the constraints of gravitational forces. In each timestep, the core algorithm iterates through particle pairs to calculate the force on each particle and update its current position.

As part of your work in a research lab, you are developing an efficient N-body simulation for the solar system that exploits computationally fast operations on matrices to speed up runtime (good thing you're taking EECS16A!). You represent each body as a vector in 3D space:

$$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

You calculate that the position of one particular body – Earth – is updated in the following way during every timestep:

- $x[t+1] = 0.5x[t] + 0.7y[t] + 0.3z[t]$
- $y[t+1] = 0.6y[t] + 0.1z[t]$
- $z[t+1] = 0.3x[t] + 0.2y[t] + z[t]$

(a) (4 points) After one timestep, at time $t = 1$, Earth is located at $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. You want to calculate the position of Earth at $t = 0$. **Formulate this problem as a matrix-vector equation in the form** $A\vec{x} = \vec{b}$**.** You do not need to solve for Earth's position.

(b) (6 points) You have determined that Neptune's position in each timestep is updated according to the following matrix:

$$\mathbf{N} = \begin{bmatrix} 1 & 0.2 & 0 \\ 0 & -0.2 & 0.1 \\ -1 & 0 & 0.1 \end{bmatrix}$$

You let the simulation run in the background for a while, and at $t = n$, Neptune is located at $\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$. You then realize that you've forgotten to record position data since you started! **Is it possible to recover Neptune's position uniquely at** $t = n-1$**?** If it is, **use Gaussian elimination to find the inverse of N,** $\mathbf{N}^{-1}$**.**

(c) (8 points) Additionally, you have the following matrix for Pluto:

$$\begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

If Pluto is positioned at some unspecified $\begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix}$ at $t = 0$, **are there any points in $\mathbb{R}^3$ space that you cannot reach at at $t = 1$? If so, what is the subspace that Pluto can be located in?**

Note: You do not have to provide rigorous justification.

(d) (6 points) After running your simulation repeatedly, you notice that with the current update matrices you have entered, Venus and Mars are all moving within the same 2D orbital plane. You refer back to your calculations, but you notice there is a smudge obscuring one element of matrix $\mathbf{M}$:

$$\mathbf{V} = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0 & 0.7 & 0.7 \\ 0.7 & 1.1 & 0.4 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0.4 & 0.6 & 0.2 \\ -1.4 & 0 & 1.4 \\ 0.6 & m_{32} & 0.8 \end{bmatrix}$$

$\mathbf{V}$ and $\mathbf{M}$ are the update matrices for Venus and Mars respectively. **Fill in the missing matrix element (denoted by "m_{32}") in a way that would explain the behavior of these 2 planets.**

8. Proof (10 points)

You are told that a $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a conservative transition matrix. **Prove that it has an eigenvalue of λ = 1.**