University of California, Berkeley
Department of EECS

EE120: Signals and Systems (Spring 2021)

Midterm 1 Solutions

Issued: 12:15 PM, February 22, 2021 **Due:** 1:45 PM, February 22, 2021

Full Name: **SID:**

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Problem 1: (40 pt) Period of signals. Determine the period of the following continuous time ($x(t)$) or discrete time ($x[n]$) signals. For periodic signals, fill the blank with **fundamental period**, for aperiodic signals, fill the blank with "**N/A**". No proof is needed, nor will it be graded. Use the provided space as draft space.

Solution:

For example, for $x(t) = \sin(t)$, the table is:

signal	fundamental period
$x(t) = \cos(t)$	2π

signal	fundamental period	signal	fundamental period
$x(t) = \sin(3t - 1)$	$\frac{2\pi}{3}$	$x[n] = \sin[3n - 1]$	N/A
$x(t) = \sin(\frac{\pi}{3}t^2 - 1)$	N/A	$x[n] = \sin[\frac{\pi}{3}n^2 - 1]$	6
$x(t) = \sin^2(\frac{\pi}{3}t - 1)$	3	$x[n] = e^{j\frac{\pi}{8}n}\cos[\frac{\pi}{3}n]$	48
$x(t) = e^{j(\pi t - 1)}$	2	$x[n] = \cos[\pi + 0.2n]$	N/A
$x(t) = e^{t(\pi j - 1)}$	N/A	$x[n] = \sum_{k=-\infty}^{\infty} e^{-(2n-k)}u[2n-k]$	1

1. $x(t) = \sin(3t - 1)$. This is a sine function with frequency of 3.

$$T = \frac{2\pi}{3}$$

2. $x(t) = \sin(\frac{\pi}{3}t^2 - 1)$. Suppose (for contradiction), that the period is T. Then,

$$\begin{aligned} x(t+T) &= x(t) \\ \sin(\frac{\pi}{3}t^2 + \frac{2\pi}{3}tT + \frac{\pi}{3}T^2 - 1) &= \sin(\frac{\pi}{3}t^2 - 1). \end{aligned}$$

Then, there would exist an integer $m \in \mathbb{Z}$ that for all $t \in \mathbb{R}$ can make

$$\begin{aligned} \frac{2\pi}{3}tT + \frac{\pi}{3}T^2 &= m2\pi \\ T^2 + 2tT &= 6m \end{aligned}$$

Since t is continuous in $\mathbb{R}$, there does not exist an integer m that will make $T^2 + 2tT = 6m$ happen. So, this signal is not periodic.

3. $x(t) = \sin^2(\frac{\pi}{3}t - 1)$. Because $x(t) = \sin(\frac{\pi}{3}t - 1)$ has fundamental period 6, so the squared version $x(t) = \sin^2(\frac{\pi}{3}t - 1)$ has all the negative lobe flipped to be positive (sketch it out to see what this would look like). Hence, the fundamental period is decreased by a factor of 2. The period is now $T = 3$.

This fact can also be verified using Euler's formula or properties of sines.

4. $x(t) = e^{j(\pi t - 1)}$. Suppose T is the period, then

$$\begin{aligned} x(t+T) &= x(t) \\ e^{j(\pi t + \pi T - 1)} &= e^{j(\pi t - 1)}. \end{aligned}$$

The phase of complex exponential repeats every 2π. So the signal is periodic when $\pi T = 2\pi$. So period is $T = 2$.

5. $x(t) = e^{t(\pi j - 1)}$. This signal can be written as

$$x(t) = e^{t(\pi j - 1)} = e^{-t}e^{\pi j t} = e^{-t}(\cos(\pi t) + j\sin(\pi t)).$$

It is a complex oscillation with an exponential decay modulation. So, the signal is not periodic.

6. $x[n] = \sin[3n - 1]$. $\frac{2\pi}{3}$ is not a rational number, thus this DT signal is not periodic.

7. $x[n] = \sin[\frac{\pi}{3}n^2 - 1]$. Suppose N is the period. We need that N to make the following equation hold,

$$x[n+N] = x[n]$$
$$\sin[\frac{\pi}{3}n^2 + \frac{\pi}{3}N^2 + \frac{2\pi}{3}nN] = \sin[\frac{\pi}{3}n^2].$$

Then, we need to make sure that there exist an integer $m \in \mathbb{Z}$ that can make

$$\frac{\pi}{3}N^2 + \frac{2\pi}{3}nN = m2\pi$$
$$N^2 + 2Nn = 6m$$

as long as we can find an N that can make for all $n \in \mathbb{N}$, $N^2 + 2Nn$ is multiple of 6, that N will be the period. For $N = 6$, $m = 6 + 2n$, $\forall n$. Therefore, $N = 6$ is the period.

8. $x[n] = e^{j\frac{\pi}{8}n}\cos[\frac{\pi}{3}n]$. This signal can be written as

$$x[n] = e^{j\frac{\pi}{8}n}\cos[\frac{\pi}{3}n] = x_1[n]x_2[n]$$

with

$$x_1[n] = e^{j\frac{\pi}{8}n}, \text{ and } x_2[n] = \cos[\frac{\pi}{3}n]$$

The period for $x_1[n]$, $x_2[n]$ are $T_1 = 16$ and $T_2 = 6$. So for $x[n] = x_1[n]x_2[n]$, the signal value repeats every $16 \times 3 = 48$ time points.

$$T = \frac{16 \times 6}{\text{gcd}(16, 6)} = 48.$$

9. $x[n] = \cos[\pi + 0.2n]$. Suppose N is the period.

$$x[n+N] = x[n]$$
$$\cos[\pi + 0.2n + 0.2N] = \cos[\pi + 0.2n].$$

There is not any integer m that satisfies $0.2N = 2\pi m$. So, this signal is not periodic.

10. $x[n] = \sum_{k=-\infty}^{\infty} e^{-(2n-k)}u[2n-k]$. Suppose N is the period.

$$x[n+N] = \sum_{k=-\infty}^{\infty} e^{-(2n+2N-k)}u(2n+2N-k)$$

Let $m = k - 2N$,

$$x[n+N] = \sum_{k=-\infty}^{\infty} e^{-(2n+2N-k)}u(2n+2N-k) = \sum_{m=-\infty}^{\infty} e^{-(2n-m)}u[2n-m] = x[n].$$

As long as k is an integer, $x[n+N] = x[n]$. Thus the smallest positive choice of N, the fundamental period, is $N = 1$.

Problem 2: (48 pt) System properties. Fill in the table for the following systems described by input-output relationships. Specify if the system is 1) linear or not, 2) time-invariant or not, 3) memoryless or not, 4) causal or not, 5) stable or not. *If you identify the system as LTI*, determine the impulse response function. If you identify the system as non-LTI, put "N/A" for the impulse response. For each system, $x(t)$ or $x[n]$ is the input, $y(t)$ or $y[n]$ is the output.

Use "✓" to indicate **yes**, use "×" to indicate **no**, "N/A" for **not applicable**. No proof is needed, nor will it be graded. Use the provided space as draft space.

1. $y(t) = x(t+3) - x(1-t)$
2. $y(t) = x(2t)$
3. $y(t) = \cos(2x(t))$
4. $y(t) = \int_{-\infty}^{t} x(u)e^{-(t-u)}du$
5. $y[n] = \sum_{k=n}^{\infty} x[k]$
6. $y[n] = \begin{cases} (-1)^n x[n] & x[n] \geq 0 \\ 2x[n] & x[n] < 0 \end{cases}$
7. $y[n] = \max\{x[n], x[n+1], \ldots, x[\infty]\}$
8. $y[n] = x[n]x[n-1]$

Solution:

For example, for an LTI, not memoryless, non-causal, and stable system $y(t) = x(t+2)$, the table is:

system	linear	time invariant	memoryless	causal	stable	impulse response h(t) or h[n]
e.g.	✓	✓	×	×	✓	$h(t) = \delta(t+2)$

system	linear	time invariant	memoryless	causal	stable	impulse response h(t) or h[n]
1	✓	×	×	×	✓	N/A
2	✓	×	×	×	✓	N/A
3	×	✓	✓	✓	✓	N/A
4	✓	✓	×	✓	✓	$e^{-t}u(t)$
5	✓	✓	×	×	×	$u[-n]$
6	×	×	✓	✓	✓	N/A
7	×	✓	×	×	✓	N/A
8	×	✓	×	✓	✓	N/A

Linear

For a continuous time system, suppose $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are input-output pairs for this system. Consider $x(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$ as an input where $\alpha_1, \alpha_2 \in \mathcal{C}$.
For a discrete time system, suppose $(x_1[n], y_1[n])$ and $(x_2[n], y_2[n])$ are input-output pairs for this system. Consider $x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$ as an input where $\alpha_1, \alpha_2 \in \mathcal{C}$.

1. **Linear.** $y(t) = x(t+3) - x(1-t) = \alpha_1 x_1(t+3) + \alpha_2 x_2(t+3)) - \alpha_1 x_1(1-t) - \alpha_2 x_2(1-t) = \alpha_1(x_1(t+3) - x_1(1-t)) + \alpha_2(x_2(t+3)) - x_2(1-t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$.

2. **Linear.** $y(t) = x(2t) = \alpha_1 x_1(2t) + \alpha_2 x_2(2t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$.

3. **Not linear.** $y(t) = \cos(2x(t)) = \cos(2\alpha_1 x_1(t) + 2\alpha_2 x_2(t)) \neq \alpha_1 \cos(2x_1(t)) + \alpha_2 \cos(2x_2(t)) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$.

4. **Linear.** $y(t) = \int_{-\infty}^{t} x(u)e^{-(t-u)}du = \int_{-\infty}^{t} (\alpha_1 x_1(u) + \alpha_2 x_2(u))e^{-(t-u)}du =$ $\alpha_1 \int_{-\infty}^{t} x_1(u)e^{-(t-u)}du + \alpha_2 \int_{-\infty}^{t} x_2(u)e^{-(t-u)}du = \alpha_1 y_1(t) + \alpha_2 y_2(t)$.

5. **Linear.** $y[n] = \sum_{k=n}^{\infty} x[k] = \sum_{k=n}^{\infty} (\alpha_1 x_1[k] + \alpha_2 x_2[k]) = \alpha_1 \sum_{k=n}^{\infty} x_1[k] + \alpha_2 \sum_{k=n}^{\infty} x_2[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$.

6. **Not linear.** For the input $x[0] = 1$, $y[0] = 1$. If $x_1[0] = -1x[0]$, $y[0] = -2 \neq -1y[0]$. It does not satisfy scaling property. So this system is not linear.

7. **Not linear.** $y[n] = \max\{(\alpha_1 x_1[n] + \alpha_2 x_2[n]), (\alpha_1 x_1[n+1] + \alpha_2 x_2[n+1]), \ldots, (\alpha_1 x_1[\infty] + \alpha_2 x_2[\infty])\} \neq \alpha_1 \max\{x_1[n], x_1[n+1], \ldots, x_1[\infty]\} + \alpha_2 \max\{x_2[n], x_2[n+1], \ldots, x_2[\infty]\} = \alpha_1 y_1[n] + \alpha_2 y_2[n]$. To easily show that the two expressions are not equivalent, try making the αs negative.

8. **Not linear.** $y[n] = (\alpha_1 x_1[n] + \alpha_2 x_2[n])(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]) \neq \alpha_1 x_1[n] x_1[n-1] + \alpha_2 x_2[n] x_2[n-1] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$.

Time invariant

For a continuous time system, consider a new input $\hat{x}(t) = x(t-\tau), \forall \tau \in \mathbb{R}$.
For a discrete time system, consider a new input $\hat{x}[n] = x[n-N], \forall N \in \mathbb{Z}$.

1. **Not time invariant.** $\hat{y}(t) = \hat{x}(t+3) - \hat{x}(1-t) = x(t-\tau+3) - x(1-t-\tau) \neq x(t-\tau+3) - x(1-t+\tau) = y(t-\tau)$.

2. **Not time invariant.** $\hat{y}(t) = \hat{x}(2t) = x(2t-\tau) \neq x(2t-2\tau) = y(t-\tau)$.

3. **Time invariant.** $\hat{y}(t) = \cos(2\hat{x}(t))\cos(2x(t-\tau)) = y(t-\tau)$.

4. **Time invariant.** $\hat{y}(t) = \int_{-\infty}^{t} \hat{x}(u)e^{-(t-u)}du = \int_{-\infty}^{t} x(u-\tau)e^{-(t-u)}du =$ $\int_{-\infty}^{t-\tau} x(u)e^{-(t-(u+\tau))}du = \int_{-\infty}^{t-\tau} x(u)e^{-(t-u-\tau)}du = y(t-\tau)$.

5. **Time invariant.** $\hat{y}[n] = \sum_{k=n}^{\infty} \hat{x}[k] = \sum_{k=n}^{\infty} x[k-N] = \sum_{k=n-N}^{\infty} x[k] = y[n-N]$.

6. **Not time invariant.** $$\hat{y}[n] = \begin{cases} (-1)^n \hat{x}[n] & \hat{x}[n] \geq 0 \\ 2\hat{x}[n] & \hat{x}[n] < 0 \end{cases} = \begin{cases} (-1)^n x[n-N] & x[n-N] \geq 0 \\ 2x[n-N] & x[n-N] < 0 \end{cases} \neq$$ $$\begin{cases} (-1)^{n-N} x[n-N] & x[n-N] \geq 0 \\ 2x[n-N] & x[n-N] < 0 \end{cases}.$$ If N is odd, $\hat{y}[n] \neq y[n-N]$.

7. **Time invariant.** $\hat{y}[n] = \max\{\hat{x}[n], \hat{x}[n+1], \ldots, \hat{x}[\infty]\} = \max\{x[n-N], x[n-N+1], \ldots, x[\infty - N]\} = y[n-N]$.

8. **Time invariant.** $\hat{y}[n] = \hat{x}[n]\hat{x}[n-1] = x[n-N]x[n-N-1] = y[n-N]$.

Memoryless.

1. **Not memoryless**. At $t = -1$, in order to compute $y(-1)$, we need $x(2)$ which is not at the current time.

2. **Not memoryless**. At $t = 1$, in order to compute $y(1)$, we need $x(2))$ which is not at the current time.

3. **Memoryless**

4. **Not memoryless**. In order to compute $y(t)$, the system needs values of $x(t)$ from $-\infty$ to t, most of them are not at the current time.

5. **Not memoryless**. In order to compute $y[n]$, the system needs values of $x[n]$ from n to ∞, most of them are not at the current time.

6. **Memoryless**.

7. **Not memoryless**. In order to compute $y[n]$, the system needs to check values of $x[n]$ from n to ∞, most of them are not at the current time.

8. **Not memoryless**. At $n = 0$, in order to compute $y[0]$, we need $x[-1]$ which is not at the current time.

Causal

1. **Not causal**. At $t = 1$, in order to compute $y(1)$, we need the future signal $x(4)$.

2. **Not causal**. At $t = 1$, in order to compute $y(1)$, we need the future signal $x(2))$.

3. **Causal**.

4. **Causal**.

5. **Not Causal**. In order to compute $y[n]$, the system needs to check the value of $x[n]$ from n to ∞, most of which are in the future time.

6. **Causal**.

7. **Not Causal**. In order to compute $y[n]$, the system needs to check the value of $x[n]$ from n to ∞, most of them are in the future time.

8. **Causal**.

Stable

For continuous time system, consider a bounded signal $|x(t)| \leq M$ as input,
For discrete time system, consider a bounded signal $|x[n]| \leq M$ as input

1. **Stable**. $|y(t)| \leq 2M < \infty$.

2. **Stable**. $|y(t)| \leq M < \infty$.

3. **Stable**. $|y(t)| \leq 1 < \infty$.

4. **Stable**. $|y(t)| \leq Me^{-t}\int_{-\infty}^{t} e^{u} = e^{-t}Me^{t} = M < \infty$.

5. **Not stable**. $|y[n]| \leq \sum_{n}^{\infty} M = \infty$.

6. **Stable**. $|y[n]| \leq 2M < \infty$

7. **Stable**. $|y[n]| = |\max\{x[n], x[n+1], \ldots, x[\infty]\}| \leq M < \infty$.

8. **Stable**. $|y[n]| \leq M^2 < \infty$

Impulse response

Set $x[n] = \delta[n]$,

4.

$$\begin{aligned}
h(t) &= \int_{-\infty}^{t} \delta(u)e^{-(t-u)}du = e^{-t}\int_{-\infty}^{t} \delta(u)e^{u}du \\
&= \begin{cases} 0, & t < 0 \\ e^{-t}\int_{-\infty}^{t} \delta(u)e^{u}du + 0, & t > 0 \end{cases} \\
&= \begin{cases} 0, & t < 0 \\ e^{-t}\int_{-\infty}^{t} \delta(u)e^{u}du + e^{-t}\int_{t}^{\infty} \delta(u)e^{u}du, & t > 0 \end{cases} \\
&= \begin{cases} 0, & t < 0 \\ e^{-t}\int_{-\infty}^{\infty} \delta(u)e^{u}du, & t > 0 \end{cases} \\
&= \begin{cases} 0, & t < 0 \\ e^{-t} & t > 0 \end{cases} = e^{-t}u(t).
\end{aligned}$$

5. $h[n] = \sum_{k=n}^{\infty} \delta[k] = \begin{cases} 1, & n \leq 0 \\ 0, & n > 0 \end{cases} = u[-n].$

Problem 3: (80 pts) Multiple choices. Circle your choice. No proof is needed, nor will it be graded. Use the provided space as draft space.

1. What is the magnitude of $(j-1)e^{j-1}$?

   [✓ (a)] $\frac{\sqrt{2}}{e}$
   (b) $j-1$
   (c) 1
   (d) 2

2. What is the phase of e^{j-1}?

   (a) -1
   (b) j
   [✓ (c)] 1
   (d) $\sqrt{2}$

3. A memoryless system must also be

   (a) not sure, because memoryless does not imply other properties of this system
   [✓ (b)] causal
   (c) stable
   (d) noncausal

4. Which of the following systems is stable?

   (a) $y(t) = \log(x(t))$
   [✓ (b)] $y(t) = \exp(x(t))$
   (c) an LTI system with impulse response $h(t) = \sin(t)$
   (d) $y(t) = tx(t) + 1$

5. Is the LTI system with impulse response $h(t) = \exp(-t)u(t)$ stable?

   [✓ (a)] Yes

(b) No

(c) It depends on the input.

6. Compute the convolution $(\exp(-at)u(t)) * u(t))$, where $u(t)$ is the unit step function.

(a) $(\frac{1}{a} - \frac{1}{a}\exp(at))u(-t)$

(b) $(\frac{1}{a} - \frac{1}{a}\exp(-at))u(-t)$

(c) $(\frac{1}{a} - \frac{1}{a}\exp(at))u(t)$

[✓ (d)] $(\frac{1}{a} - \frac{1}{a}\exp(-at))u(t)$

7. Compute the convolution $h[n] * \delta[n-5]$.

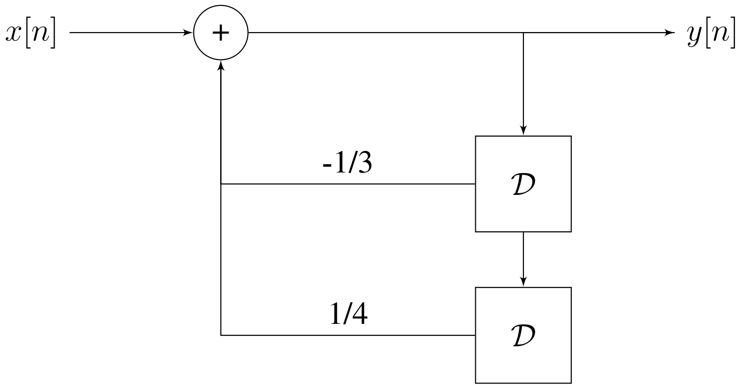
(a) $h[n+5]$

[✓ (b)] $h[n-5]$

(c) $h[5]$

(d) $h[-5]$

8. Write an LCCDE characterizing the following system:



(a) $y[n] = x[n] - \frac{1}{3}x[n-1] + \frac{1}{4}x[n-1]$

(b) $y[n] = x[n] - \frac{1}{3}x[n-1] + \frac{1}{4}x[n-2]$

(c) $y[n] = x[n] - \frac{1}{3}y[n-1] + \frac{1}{4}y[n-1]$

[✓ (d)] $y[n] = x[n] - \frac{1}{3}y[n-1] + \frac{1}{4}y[n-2]$

9. What's the Fourier transform of $\text{rect}(t) = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$?

(a) $\sin(\pi f)$

(b) $\sin(\omega)$

(c) $\frac{\sin(\omega)}{\omega}$

[✓ (d)] $\frac{\sin(\pi f)}{\pi f}$

10. What's the Fourier transform of $\cos(\omega_0 t + \phi)$?

(a) $\sin(\omega t + \phi)$

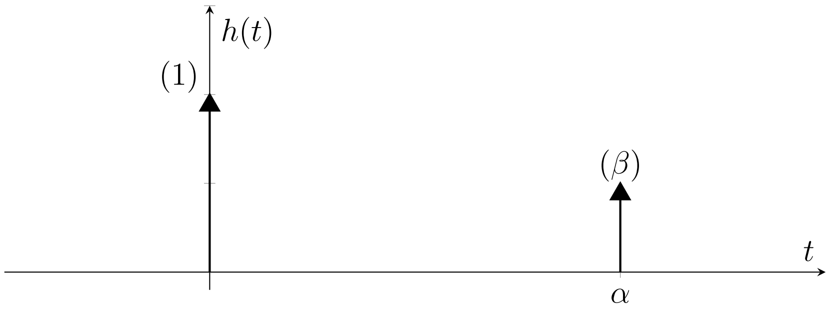
(b) $\cos(\pi f)$

[✓ (c)] $\pi e^{i\phi}\delta(\omega - \omega_0) + \pi e^{-i\phi}\delta(\omega + \omega_0)$

(d) $\pi e^{i\phi}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$

Problem 4: (72 pt) Echoed Signal. You're recording sound in a room with an echo, so your microphone picks up both the original signal and a delayed, attenuated version of the signal. You decide to model this process as an LTI system with impulse response

$$h(t) = \delta(t) + \beta\delta(t - \alpha).$$



In this problem, we will examine the behavior of this system in the frequency domain.

1. (10 pt) Find the Fourier Transform of the impulse response, $H(\omega)$.

Solution:

Apply the CTFT analysis equation:

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty} h(t)e^{-i\omega t}\, dt \\ &= \int_{-\infty}^{\infty} \big(\delta(t) + \beta\delta(t-\alpha)\big)e^{-i\omega t}\, dt \\ &= 1 + \beta e^{-i\omega\alpha}. \end{aligned}$$

2. (12 pt) For $\alpha = 2$ and $\beta = 1$, find expressions for and plot the magnitude and phase of $H(\omega)$. Your expressions should be closed-form, but piecewise expressions are allowed.

Solution:
Plugging in the given values for α and β, we have

$$H(\omega) = 1 + e^{-i2\omega}.$$

Factor out an $e^{-i\omega}$ to get

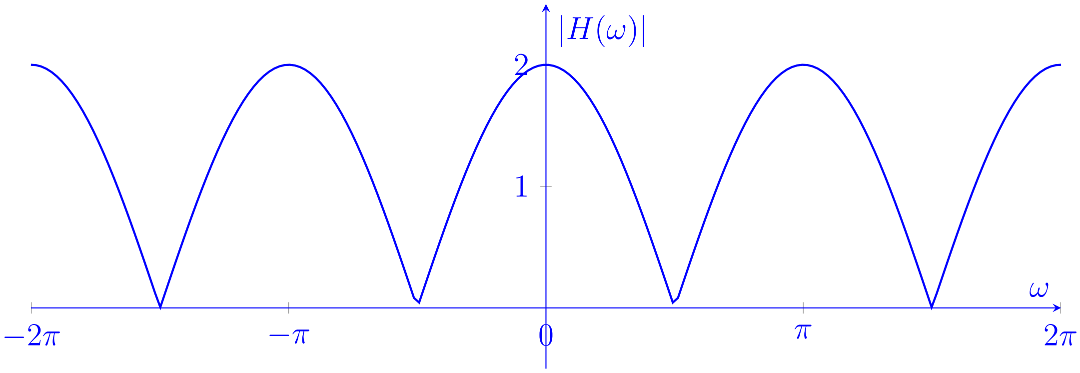
$$H(\omega) = e^{-i\omega}(e^{i\omega} + e^{-i\omega}) = 2e^{-i\omega}\cos(\omega).$$

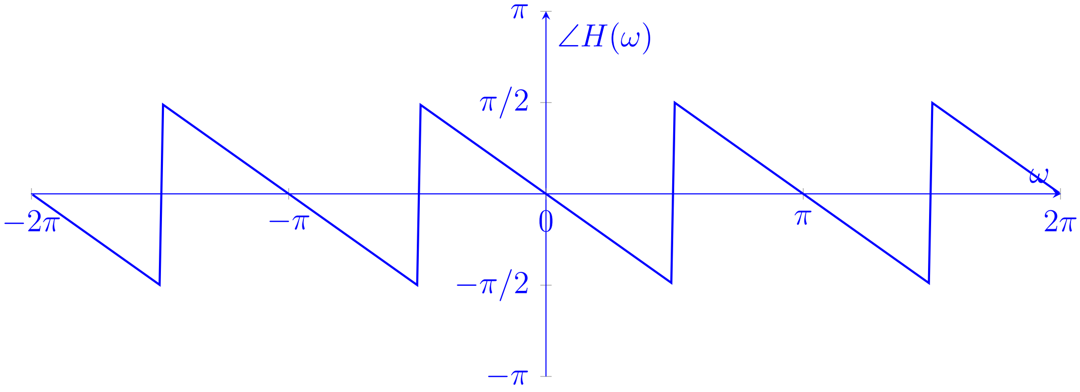
Take the magnitude and phase to get

$$|H(\omega)| = |2\cos(\omega)|.$$
$$\angle H(\omega) = \angle e^{-i\omega} + \angle 2\cos(\omega).$$

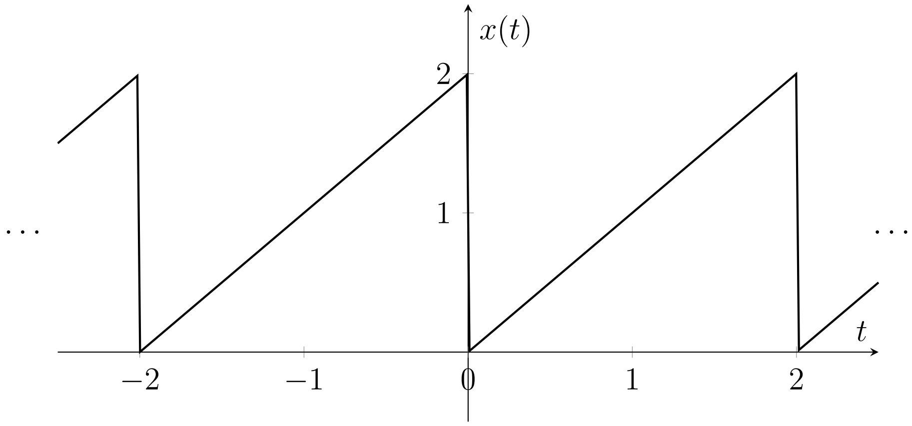
The phase of $e^{-i\omega}$ is $-\omega$, and the phase of $2\cos(\omega)$ is 0 when $\cos$ is positive and π when $\cos$ is negative. So,

$$\angle H(\omega) = -\omega + \begin{cases} 0, & \omega \in [-\frac{\pi}{2}, \frac{\pi}{2}] + 2k\pi; \\ \pi, & \omega \in [\frac{\pi}{2}, -\frac{\pi}{2}] + 2k\pi \end{cases} \quad \text{where } k \in \mathbb{Z}.$$





3. (12 pt) Let the input signal, $x(t)$ be a triangle wave, as pictured below. **Note that all subsequent parts of this question will use this definition of** $x(t)$.



$x(t) = t, \forall t \in [0, 2]$, and $x(t)$ repeats with a period of 2.

Find the CTFS coefficients a_k in the complex exponential representation of $x(t)$.

Solution:

The signal has a period of 2, so the fundamental frequency is $\omega_0 = \frac{2\pi}{2} = \pi$.

$$a_k = \frac{1}{2}\int_0^2 x(t)e^{-i\pi kt}\,dt.$$

For $k = 0$,

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^2 te^0\,dt \\ &= \frac{1}{2}\int_0^2 t\,dt = 1. \end{aligned}$$

For $k \neq 0$, we can use integration by parts, choosing $u = t$ and $dv = e^{-i\pi kt}\,dt$.

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^2 te^{-i\pi kt}\,dt \\ &= \frac{-t}{2i\pi k}e^{-i\pi kt}\Big|_0^2 + \frac{1}{2i\pi k}\int_0^2 e^{-i\pi kt}\,dt \\ &= \frac{-1}{i\pi k}e^{-i2\pi k} - \frac{1}{2(i\pi k)^2}e^{-i\pi k}\Big|_0^2 \end{aligned}$$

For integer k, $e^{-i2\pi k} = 1$, so

$$a_k = \frac{-1}{i\pi k} - \frac{1}{2(i\pi k)^2}(1-1) = -\frac{1}{i\pi k}.$$

4. (12 pt) Show that, if the CTFS coefficients of an arbitrary signal $g(t)$ are represented by b_k in complex exponential representation, then the CTFT of $g(t)$ is

$$G(\omega) = A \sum_k b_k \delta(\omega - k\omega_0),$$

and find a value for A.

Hint: start with the inverse CTFT of $g(t)$. *What does the resulting expression look like?*

Solution:

Following the hint, plug in the given expression for $G(\omega)$ into the inverse CTFT:

$$\begin{aligned} g(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{i\omega t}\,d\omega \\ &= \frac{A}{2\pi}\int_{-\infty}^{\infty}\left(\sum_k b_k\delta(\omega - k\omega_0)\right)e^{i\omega t}\,d\omega \\ &= \frac{A}{2\pi}\int_{-\infty}^{\infty}\left(\sum_k b_k e^{i\omega t}\delta(\omega - k\omega_0)\right)d\omega \end{aligned}$$

Applying the sifting property of the Dirac delta,

$$\begin{aligned} g(t) &= \frac{A}{2\pi}\int_{-\infty}^{\infty}\left(\sum_k b_k e^{ik\omega_0 t}\delta(\omega - k\omega_0)\right)d\omega \\ &= \frac{A}{2\pi}\sum_k b_k e^{ik\omega_0 t}. \end{aligned}$$

The CTFS expression for $g(t)$ is

$$g(t) = \sum_k b_k e^{ik\omega_0 t},$$

which exactly matches the above expression for $g(t)$ when $A = 2\pi$.

5. (12 pt) Find the CTFT $X(\omega)$ of the input signal $x(t)$ defined in part 3. Find an expression for and plot (with carefully labeled ticks on both axes) the magnitude $|X(\omega)|$ in the range $\omega \in [-4\pi, 4\pi]$. Leave your expressions in summation form.

Hint: the magnitude of a series of delta functions with different shifts can be written as

$$\left|\sum_i A_i\delta(t - T_i)\right| = \sum_i |A_i|\delta(t - T_i).$$

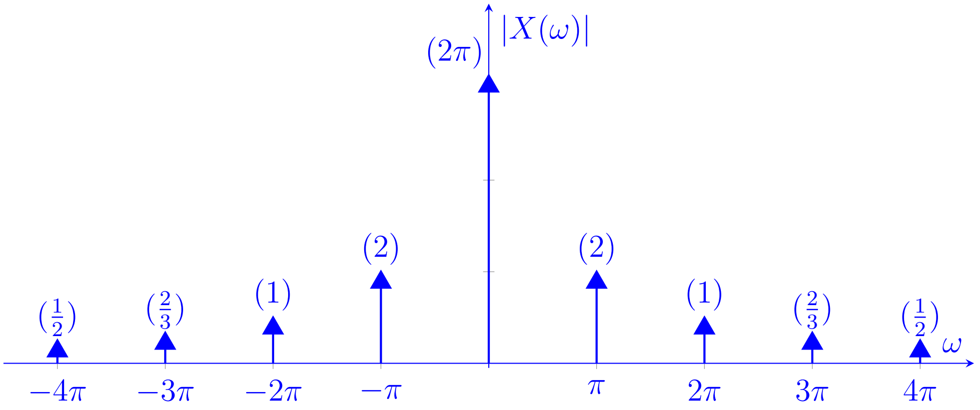
Solution:

Using the equation relating the CTFS and CTFT from the previous part, we have

$$X(\omega) = 2\pi \sum_k a_k \delta(\omega - k\pi) = 2\pi\delta(\omega) - 2\sum_{k\neq 0} \frac{1}{ik}\delta(\omega - k\pi).$$

Following the hint, the magnitude of $X(\omega)$ is

$$\begin{aligned}|X(\omega)| &= |2\pi|\delta(\omega) + \sum_{k\neq 0}\left|-\frac{2}{ik}\right|\delta(\omega - k\pi)\\ &= 2\pi\delta(\omega) + \sum_{k\neq 0}\frac{2}{|k|}\delta(\omega - k\pi).\end{aligned}$$



6. (14 pt) If $y(t) = x(t) * h(t)$, find the CTFT of $y(t)$, $Y(\omega)$.
Find an expression for and plot the magnitude $|Y(\omega)|$ in the interval $\omega \in [-4\pi, 4\pi]$ for the following values of α and β:
   (a) $\alpha = 3$, $\beta = 1/2$.
   (b) $\alpha = 2$, $\beta = 1/2$.

Solution:

Convolution in time corresponds to multiplication in frequency, so

$$\begin{aligned}Y(\omega) = H(\omega)X(\omega) &= 2\pi\big(1 + \beta e^{i\omega\alpha}\big)\delta(\omega) - 2\big(1 + \beta e^{-i\omega\alpha}\big)\sum_{k\neq 0}\frac{1}{ik}\delta(\omega - k\pi)\\ &= 2\pi(1+\beta)\delta(\omega) - 2\sum_{k\neq 0}\frac{1}{ik}\big(1 + \beta e^{-i\pi k\alpha}\big)\delta(\omega - k\pi).\end{aligned}$$

As in the previous part, we can take the magnitude of $Y(\omega)$ to get

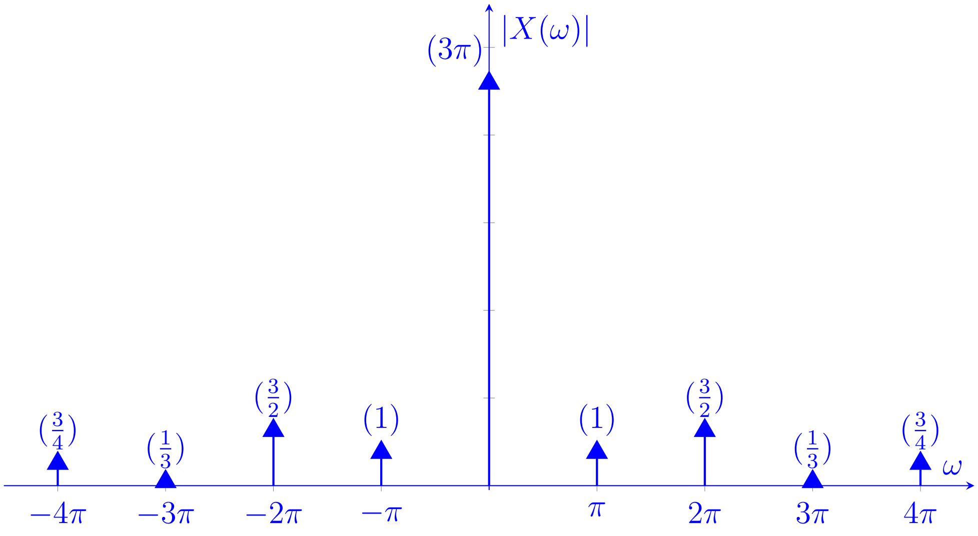
$$|Y(\omega)| = 2\pi|1+\beta|\delta(\omega) + 2\sum_{k\neq 0}\left|\frac{1}{ik}\left(1+\beta e^{-i\pi k\alpha}\right)\right|\delta(\omega - k\pi).$$

(a) For $\alpha = 3$ and $\beta = 1/2$,

$$|Y(\omega)| = 3\pi\delta(\omega) + 2\sum_{k\neq 0}\frac{1}{|k|}\left|1+\frac{1}{2}e^{-i3\pi k}\right|\delta(\omega - k\pi).$$

$e^{-i3\pi k}$ is 1 for even k and -1 for odd k, so we can write out the magnitude as

$$|Y(\omega)| = 3\pi\delta(\omega) + \sum_{k\neq 0}\frac{3}{|2k|}\delta(\omega - 2k\pi) + \sum_{k}\frac{1}{|2k+1|}\delta(\omega - (2k+1)\pi).$$

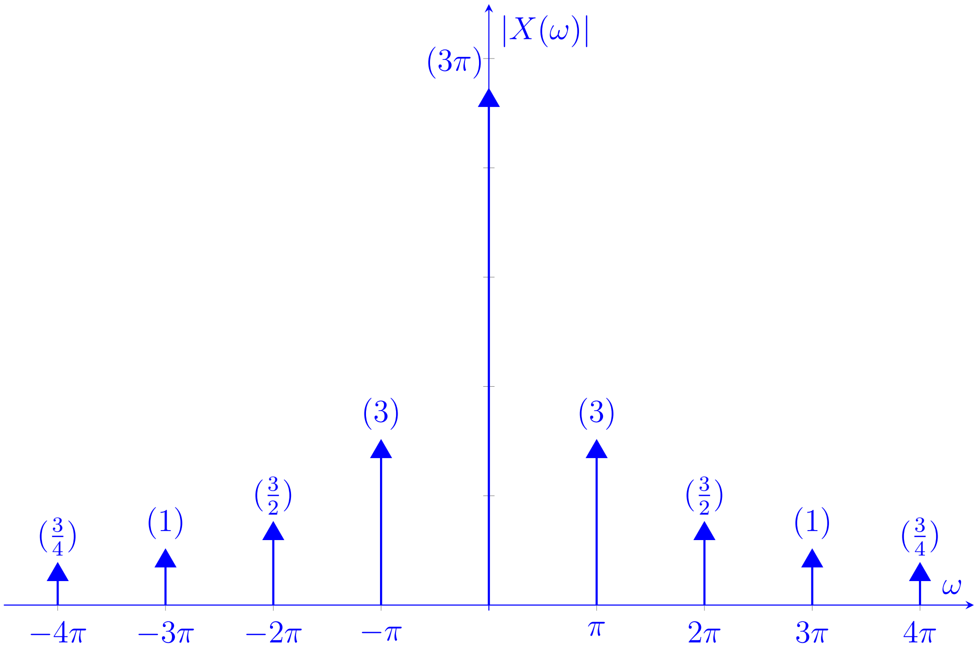


(b) For $\alpha = 2$ and $\beta = 1/2$,

$$|Y(\omega)| = \pi\delta(\omega) + 2\sum_{k\neq 0}\frac{1}{|k|}\left|1+\frac{1}{2}e^{-i2\pi k}\right|\delta(\omega - k\pi).$$

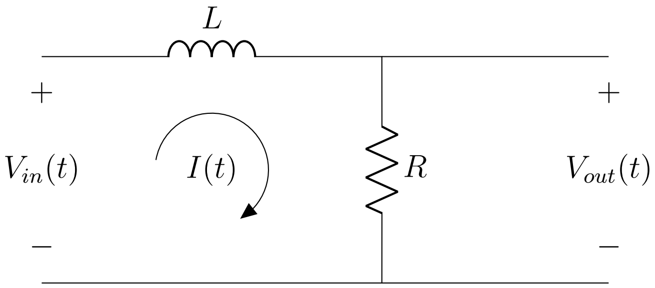
$e^{-i2\pi k}$ is always 1, so we can write out the magnitude as

$$|Y(\omega)| = \pi\delta(\omega) + \sum_{k\neq 0}\frac{3}{|k|}\delta(\omega - k\pi).$$



Problem 5: (60 pt) RL circuits.

Consider the following RL circuit



The system can be represented by the equation

$$RI(t) + L\frac{dI(t)}{dt} = V_{in}(t).$$

$V_{in}(t)$ is the input, and the voltage across the resistor is the output, $V_{out}(t) = V_R(t) = RI(t)$. The system is causal and starts at rest.

1. (14 pt) Find the impulse response $h(t)$ for the voltage across the resistor. Assume the current through the resistor is of the form $Ae^{st}u(t)$.

Solution:

Substitute $I(t) = Ae^{st}u(t)$ into the LCCDE, and set right hand side $x(t) = \delta(t)$, we have

$$RAe^{st}u(t) + LAse^{st}u(t) + LAe^{st}\delta(t) = \delta(t).$$

The right-hand side is 0 when $t \neq 0$, so we must make the left-hand side 0 for nonzero t as well. Since $A \neq 0$ and $e^{st} \neq 0$, in order to make the factors for $u(t)$ to cancel out, we must choose s such that

$$RAe^{st}u(t) + LAse^{st}u(t) = 0.$$

So, $s = -\frac{R}{L}$. The terms containing $\delta(t)$ must match as well:

$$LAe^{-\frac{R}{L}t}\delta(t) = \delta(t).$$

At $t = 0, \delta(0) \neq 0$, so we can remove $\delta(t)$ from both sides, at $t = 0$ then we have

$$LA = 1,$$

so

$$A = \frac{1}{L}.$$

In total we have $I(t) = \frac{1}{L}e^{-\frac{R}{L}t}u(t)$. Now the output voltage across the resistor when the input is $\delta(t)$ will be

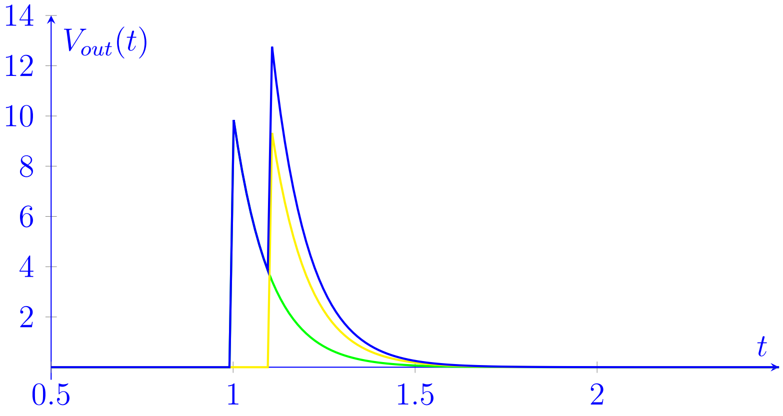
$$h(t) = RI(t) = \frac{R}{L}e^{-\frac{R}{L}t}u(t).$$

2. (12 pt) Assume the system is at rest, and at $t = 1$ s and $t = 1.1$ s, it receives two sudden impulse shocks from V_{in}. This can be modeled as two impulse inputs as $V_{in}(t) = \delta(t-1) + \delta(t-1.1)$. Calculate $V_{out}(t)$ for input $V_{in}(t) = \delta(t-1) + \delta(t-1.1)$ and sketch $V_{out}(t)$.
For sketching, you can use the following set of parameters: $R = 1$ mΩ, and $L = 100$ uH, label the location and heights of peaks.

Solution:
$V_{in}(t) = \delta(t-1) + \delta(t-1.1)$, so

$$\begin{aligned} V_{out}(t) &= V_{in}(t) * h(t) \\ &= (\delta(t-1) + \delta(t-1.1)) * \frac{R}{L}e^{-\frac{R}{L}t}u(t) \\ &= \frac{R}{L}e^{-\frac{R}{L}(t-1)}u(t-1) + \frac{R}{L}e^{-\frac{R}{L}(t-1.1)}u(t-1.1) \end{aligned}$$

For $R = 1$ mOhm, and $L = 100$ uH, $\frac{R}{L} = 10$ s^{-1}



The two terms of the expression for $V_{out}(t)$ are shown in green and yellow, respectively.

3. (10 pt) Find the frequency response $H(\omega)$ for the voltage on the resistor.

Solution:

Frequency response $H(\omega)$ is the CTFT of $h(t)$.

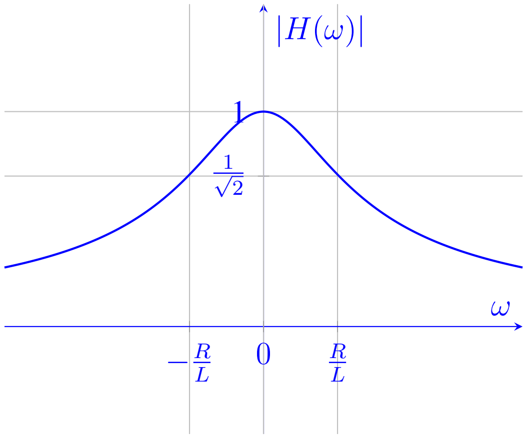
$$\begin{aligned}\mathcal{F}\{h(t)\} &= \int_{-\infty}^{\infty} \frac{R}{L} e^{-\frac{R}{L}t} u(t) e^{-j\omega t} dt \\ &= \int_{0}^{\infty} \frac{R}{L} e^{-\frac{R}{L}t} e^{-j\omega t} dt \\ &= -\frac{R}{L} \frac{1}{\frac{R}{L} + j\omega} e^{-(\frac{R}{L}+j\omega)t} \Big|_0^{\infty} \\ &= \frac{R}{R + j\omega L} = H(\omega).\end{aligned}$$

4. (8 pt) Sketch a well-labeled magnitude plot of the frequency response.

Solution:

$$|H(\omega)| = |\frac{R}{R + j\omega L}| = \frac{R}{\sqrt{R^2 + (L\omega)^2}}.$$

It's a even function of ω, and $|H(\omega)|$ take the maximum value at $\omega = 0$.



5. (16 pt) Find the output when the input is $V_{in}(t) = e^{-t}u(t)$.

Solution:

In order to find the output $V_{out}(t)$, we need to solve the equation for the current of the circuit. Assuming the homogeneous solution of the current $y_h(t)$ is of the form

$$y_h(t) = Ae^{st}, A \neq 0.$$

Substitute $y_h(t)$ into the LCCDE, and set the right hand side to $x(t) = 0$.

$$RAe^{st} + LAse^{st} = 0.$$

Since $A \neq 0, e^{st} \neq 0$, we must have $s = -\frac{R}{L}$ to make the left-hand side 0. So the homogeneous solution of the current is

$$y_h(t) = Ae^{-\frac{R}{L}t}.$$

Assume the particular solution of the current evolution $y_p(t)$ is of the form

$$y_p(t) = Ke^{bt}, K \neq 0, \forall t > 0.$$

Substitute $y_p(t)$ into LCCDE, and set the right hand side to $x(t) = e^{-t}$. So, for $t > 0$,

$$RKe^{bt} + LKbe^{bt} = e^{-t}.$$

Comparing both sides, we have

$$b = -1,$$

$$RK - LK = 1, \text{ so } K = \frac{1}{R - L}.$$

Thus, we have

$$y_p(t) = \frac{1}{R-L}e^{-t}u(t).$$

$I(t) = y_h(t) + y_p(t) = Ae^{-\frac{R}{L}t} + \frac{1}{R-L}e^{-t}u(t)$. Since the system starts at rest, for $t < 0$, $y(t) = 0$, and when $t = 0$, $y(0) = A + \frac{1}{R-L} = 0$. Thus, $A = \frac{1}{L-R}$, and

$$y(t) = \frac{1}{L-R}(e^{-\frac{R}{L}t} - e^{-t})u(t).$$

So the output when input is $V_{in}(t) = e^{-t}u(t)$ is

$$V_{out}(t) = RI(t) = \frac{R}{L-R}(e^{-\frac{R}{L}t} - e^{-t})u(t)$$