S20 PHYSICS 7B: Bordel Final Solutions

Friendly neighborhood 7B GSIs

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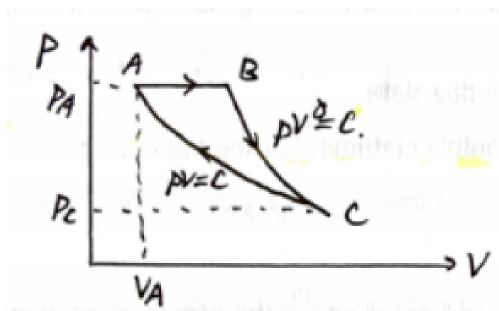
1 Problem 1

(a)



As a linear gas molecule with negligible vibrational energy, there are 5 degrees of freedom, and hence $\gamma = \frac{7}{5} \approx \frac{3}{2}$.

The gas starts at P_A and V_A. Then the ideal gas law gives the temperature as

$$T_A = \frac{P_A V_A}{nR}. \tag{1.1}$$

After isobaric expansion to B, by definition, $P_B = P_A$. At this point, we don't know V_B and hence don't know T_B, so let's consider the next process. After the adiabatic expansion, the gas is at P_C and V_C, where P_C is given. Because the process is adiabatic, we have $P_B V_B^{\gamma} = P_C V_C^{\gamma}$.

We don't know V_B yet, so let's consider the final process: an isothermal compression from V_C to V_A. Because the compression is isothermal, we must have that $T_C = T_A$. Then applying the ideal gas law, we find

$$V_C = \frac{nRT_A}{P_C} = \frac{P_A}{P_C} V_A. \tag{1.2}$$

Then we can now determine V_B using the adiabatic condition:

$$V_B = \left(\frac{P_C}{P_B}\right)^{1/\gamma} V_C = \left(\frac{P_C}{P_A}\right)^{2/3} \frac{P_A}{P_C} V_A = \left(\frac{P_A}{P_C}\right)^{1/3} V_A. \tag{1.3}$$

And correspondingly the temperature:

$$T_B = \frac{P_B V_B}{nR} = \left(\frac{P_A}{P_C}\right)^{1/3} \frac{P_A V_A}{nR} \tag{1.4}$$

(b)

We want to determine the heat in each process. For $A \to B$, we have $Q_{AB} = nC_P\Delta T = \frac{7}{2}nR(T_B - T_A) = \frac{7}{2}\left[\left(\frac{P_A}{P_C}\right)^{1/3} - 1\right] P_A V_A > 0$. In the process $B \to C$, we have $Q_{BC} = 0$. In the process $B \to C$, we have $Q_{CA} = nRT_A \ln\left(\frac{V_A}{V_C}\right) = nRT_A \ln\left(\frac{P_C}{P_A}\right) = P_A V_A \ln\left(\frac{P_C}{P_A}\right) < 0$. Then the efficiency is

$$\eta = \frac{Q_{\text{net}}}{Q_{\text{in}}} = \frac{\frac{7}{2}\left[\left(\frac{P_A}{P_C}\right)^{1/3} - 1\right] + \ln\left(\frac{P_C}{P_A}\right)}{\frac{7}{2}\left[\left(\frac{P_A}{P_C}\right)^{1/3} - 1\right]}. \tag{1.5}$$

(c)

Plugging in $P_A = 8P_C$:

$$\eta = 1 - \frac{2\ln(8)}{7\left[(8)^{1/3} - 1\right]} = 1 - \frac{6}{7}\ln 2 \sim 1 - \frac{6}{10} = \frac{2}{5} \tag{1.6}$$

The ideal Carnot efficiency is given by

$$\eta_C = 1 - \frac{T_{\text{cold}}}{T_{\text{hot}}} = 1 - \frac{T_A}{T_B} = 1 - \left(\frac{P_C}{P_A}\right)^{1/3} = \frac{1}{2}. \tag{1.7}$$

So $\eta_C \geq \eta$, as expected.

2 Problem 2

(a)

The electric field should be perpendicular to the conducting plane. Therefore, the potential should be constant very near the surface, and hence $V = 0$. The perpendicular condition then requires $E_x = E_y = 0$, while $E_z = \frac{\sigma}{\epsilon_0}$ is just the electric field from a charged conductor.

(b)

We should place a point charge of charge $-q$ at $x = y = 0$ and $z = -d$.

(c)

The potential will just be the superposition of the two point charges:

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right). \tag{2.1}$$

(d)

The technique we will use is to determine the electric field near $z = 0$, and then use that to determine σ. Evidently, we need only consider E_z, so:

$$E_z(x, y, 0) = -\left.\frac{\partial V}{\partial z}\right|_{z=0} \tag{2.2}$$
$$= -\frac{q}{4\pi\epsilon_0}\left(\frac{z+d}{(x^2+y^2+(z+d)^2)^{3/2}} - \frac{z-d}{(x^2+y^2+(z-d)^2)^{3/2}}\right)_{z=0} \tag{2.3}$$
$$= -\frac{q}{2\pi\epsilon_0}\frac{d}{(x^2+y^2+d^2)^{3/2}} \tag{2.4}$$
$$= \frac{\sigma}{\epsilon_0}. \tag{2.5}$$

And we hence find

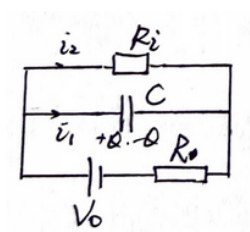
$$\sigma = -\frac{qd}{2\pi(x^2+y^2+d^2)^{3/2}}. \tag{2.6}$$

3 Problem 3

(a)

Replace the capacitor with a capacitor and a resistor in parallel.

(b)



We apply Kirchhoff's loop law to the loop created by resistor R and the capacitor. We get

$$V_0 - V_C - (I_1 + I_2)R = 0, \tag{3.1}$$

where $V_C = \frac{Q}{C}$ is the voltage drop of the capacitor and Q is the charge on the capacitor. We therefore have $I_1 = \dot{Q}$. Consider now the loop formed within the parallel circuit:

$$I_2 R_i - V_C = 0, \tag{3.2}$$

so $I_2 = \frac{V_C}{R_i} = \frac{Q}{CR_i}$. The first loop equation now reads:

$$V_0 - \frac{Q}{C} - \left(\dot{Q} + \frac{Q}{CR_i}\right)R = 0. \tag{3.3}$$

Rewriting into standard form:

$$\dot{Q} + Q\frac{R+R_i}{CRR_i} - \frac{V_0}{R} = 0, \tag{3.4}$$

which we can solve with the initial condition $Q(0) = 0$:

$$Q(t) = V_0 C \frac{R_i}{R + R_i} \left(1 - e^{-t/\tau}\right), \tag{3.5}$$

where $\tau = \frac{RR_i}{R+R_i} C$.

(c)

The maximum charge is obtained for $t \to \infty$, from which we see $Q \to CV_0 \frac{R_i}{R+R_i}$. We saw the time constant τ has a dependence on R_i as $\tau = \frac{RR_i}{R+R_i} C$. This is nothing more than replacing the value of R in the standard RC circuit with the equivalent resistance of the internal resistor and series resistor in parallel. Note that $\frac{RR_i}{R+R_i} \leq R$, so the time constant has been reduced, and the capacitor will charge more quickly.

(d)

Consider

$$\dot{Q}(0) = V_0 C \frac{R_i}{R + R_i} \frac{1}{\tau} = \frac{V_0}{R}. \tag{3.6}$$

The ideal limit of the capacitor is given by taking $R_i \to \infty$, from which we obtain the standard RC circuit equation:

$$Q(t) = V_0 C \left(1 - e^{-t/\tau}\right), \tag{3.7}$$

with $\tau = RC$. We then see

$$\dot{Q}(0) = \frac{V_0 C}{\tau} = \frac{V_0}{R}, \tag{3.8}$$

and we hence find that the initial charging rate in each case is the same.

4 Problem 4

(a)

Let us first determine the magnetic field on the symmetry axis of a ring of radius r with current I. The Biot-Savart law says

$$dB_z = \frac{\mu_0 I}{4\pi} \frac{d\ell}{r^2 + z^2} \sin\theta = \frac{\mu_0 I}{4\pi} \frac{r\, d\ell}{(r^2 + z^2)^{3/2}}, \tag{4.1}$$

and integrating around the ring then gives

$$B_z = \frac{\mu_0 I}{2} \frac{r^2}{(r^2 + z^2)^{3/2}}. \tag{4.2}$$

Now consider the ribbon as being constructed from several such rings layered on each other. That is, each ring has an infinitesimal current dI running through it, where $dI = J\, da$, where J is the "current density," or the current that passes through a cross-section of the ribbon da. Because the current is uniformly distributed, we have $J = \frac{I}{wt}$. Then we have:

$$dB_z = \frac{\mu_0\, dI}{2} \frac{r^2}{(r^2 + z^2)^{3/2}} = \frac{\mu_0 J}{2} \frac{r^2}{(r^2 + z^2)^{3/2}} da = \frac{r^2}{(r^2 + z^2)^{3/2}} dr\, dz', \tag{4.3}$$

where the z'-direction is the same as the z-direction. Then we integrate this to obtain the final magnetic field

$$B_z = \int_{R_1}^{R_2} \int_{-t/2}^{t/2} \frac{\mu_0 J}{2} \frac{wtr^2}{(r^2+z^2)^{3/2}} dz'\, dr = \frac{\mu_0 I}{2w} \int_{R_1}^{R_2} \frac{r^2}{(r^2+z^2)^{3/2}} dr. \tag{4.4}$$

By symmetry, this is the only non-zero component of the magnetic field.

(b)

The applied magnetic field will act on the moving charges, causing them to build up on one side of the ring. The build-up will generate an electric field on the conductor that will stabilize the charges against the magnetic force.

(c)

We can directly compute the drift velocity of the free charges:

$$v_d = \frac{I}{enA} = \frac{I}{enwt}. \tag{4.5}$$

The magnetic force magnitude on these charges is then

$$F_B = ev_d B = \frac{IB}{nwt}. \tag{4.6}$$

In equilibrium, this is equal in magnitude to the electric force:

$$F_E = \frac{IB}{nwt} = eE. \tag{4.7}$$

Then we can immediately compute the Hall voltage from the electric field:

$$V_H = wE = \frac{IB}{net}. \tag{4.8}$$

By the RHR, the (negative) charges are gathering on the outside edge of the ring, and hence the inner edge of the ring will have a higher potential.

(d)

We showed above:

$$V_H = wE = \frac{w}{e} F_E = \frac{w}{e} F_B = wv_d B, \tag{4.9}$$

and hence

$$v_d = \frac{V_H}{wB} \tag{4.10}$$

5 Problem 5

(a)

The current in the loop is due to the time-dependence of the current density j_s. The current density generates a magnetic field that has a non-zero flux through the loop. Because j_s is time-dependent, so too is the flux, and hence by Faraday's law there is a current generated.

The sheet generates a field pointing out of the page. By Lenz's Law, a CCW current would be generated due to the flux of the magnetic field decreasing.

(b)

The induced emf obeys $\mathcal{E} = IR$, and is also given by Faraday's law $\mathcal{E} = -\frac{d\Phi_B}{dt}$. We can write this:

$$-\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} = IR = ab\beta B_0 = \int \beta B_0 \hat{x} \cdot d\vec{a}, \tag{5.1}$$

where $\hat{x}$ points out of the page. In particular, we are careful to note that the integration with respect to $\vec{a}$ is the same $\vec{a}$ on both sides – it is the area element of the loop pointing in the $\hat{x}$ direction, and we know that $\vec{B} \propto \hat{x}$. Then we can take the derivative with respect to $\vec{a}$ and match magnitudes:

$$-\frac{\partial B}{\partial t} = \beta B_0. \tag{5.2}$$

This can then be solved with the initial condition $B(0) = B_0$:

$$\vec{B}(t) = B_0(1 - \beta t)\hat{x}. \tag{5.3}$$

(c)

Use Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}. \tag{5.4}$$

Choose an Amperian loop that is rectangular, oriented such that the current density passes through its enclosed surface, of height $2z$, and of width ℓ. We can see that only 2 legs of the loop (pointing along $\pm\hat{z}$) will contribute to the integral. The enclosed current is then just $j_s\ell$. Then Ampere's law gives

$$\oint \vec{B} \cdot d\vec{l} = 2\ell B_0(1 - \beta t) = \mu_0 j_s \ell, \tag{5.5}$$

so

$$\vec{B}(t) = B_0(1 - \beta t)\hat{z} = \frac{\mu_0 j_s}{2}\hat{z}. \tag{5.6}$$

We can see that there is no relationship between j_s and $B(t)$, and hence, the height h is irrelevant in the problem.

(d)

We use our work from part (c):

$$j_s(t) = \frac{2B(t)}{\mu_0} = \frac{2B_0}{\mu_0}(1 - \beta t) \tag{5.7}$$

6 Problem 6

(a)

At an infinitesimal section $r \to r + dr$, the amount of charge is given by $dQ = \lambda(r)\, dr = \lambda_0 \left(1 - \frac{r}{L}\right) dr$. The velocity of the segment is $v = \omega_0 r$, and we hence find that for a ring at radius r and circumference $\ell = 2\pi r$:

$$dI\, \ell = dQ\, v = \lambda_0 \omega_0 r \left(1 - \frac{r}{L}\right) dr. \tag{6.1}$$

We hence find

$$dI = \frac{\lambda_0 \omega_0}{2\pi} \left(1 - \frac{r}{L}\right) dr. \tag{6.2}$$

(b)

The infinitesimal magnetic dipole moment of the rod is

$$d\vec{\mu} = dI(r)A(r) = \frac{\lambda_0\omega_0}{2\pi}\left(1 - \frac{r}{L}\right)\pi r^2\, dr\hat{z}, \tag{6.3}$$

where z points in along the direction of the angular velocity vector $\vec{\omega}$ as determined by the RHR. Then

$$\vec{\mu} = \frac{\lambda_0\omega_0}{2}\int_0^L \left(1 - \frac{r}{L}\right) r^2\, dr\,\hat{z} = \frac{\lambda_0\omega_0}{2}\left(\frac{L^3}{3} - \frac{L^3}{4}\right)\hat{z} = \frac{\lambda_0\omega_0 L^3}{24}\hat{z}. \tag{6.4}$$

(c)

The dipole moment is given by

$$\vec{\mu} = I\vec{A} = I\pi R^2(\cos\theta\hat{x} + \sin\theta\hat{y}), \tag{6.5}$$

where x points in the direction of the magnetic field and y is the perpendicular direction.

The torque of the dipole is

$$\hat{\tau} = \vec{\mu}\times\vec{B} = -IB\pi R^2\sin\theta\,\hat{x}. \tag{6.6}$$

Setting this equal to the mechanical torque:

$$I_m\ddot{\theta} = -IB\pi R^2\sin\theta, \tag{6.7}$$

where I_m is the moment of inertia about the symmetry axis $I_m = mR^2/2$, so we get an equation of motion

$$\ddot{\theta} = -\frac{2IB\pi}{m}\sin\theta. \tag{6.8}$$

(d)

Taking θ small, the equation of motion becomes

$$\ddot{\theta} = -\frac{2IB\pi}{m}\theta, \tag{6.9}$$

which is precisely the equation of motion for a harmonic oscillator with frequency $\omega = \sqrt{\frac{2IB\pi}{m}}$.