Physics 89 Final Exam
Friday, 12/18/2020, 8am – 12/19/2020, 8am (PT) [24 hours]

Instructions

1. Please submit the exam using **Gradescope**.
2. Please solve all 6 problems below.
3. **You are required to work on this problem set on your own. In particular, collaboration or consultation with others is not permitted.**
4. You are **allowed** to look for guidance in our textbook, as well as any other books, or online sources, as long as you work alone. For example, you are allowed to check your answers with `Mathematica's` online calculator at:
`https://www.wolframalpha.com/calculators/integral-calculator`.
However, you are not allowed to seek advice on help forums.
5. Posting *questions* on our **Piazza** webpage is allowed, but please **do not answer** such questions. The instructors will reply, if an answer can be given without divulging the solution.
6. You have **24 hours** to work on the problem set. DSP accommodations allow you extra time as usual. Please also let us know if you have other major time commitments or other obstacles during this period.
7. The maximal score is 110.
8. Your total grade X (even if it is > 100) will go into the formula for the final grade in the course

$$\begin{aligned}\text{Final Grade} &= 0.2Y + Z \\ Y &= \text{Sum of best 10 out of 12 problem sets.} \\ Z &= \text{Max}(0.4\,[\text{Midterm}] + 0.4W,\ 0.8W) \\ W &= \text{Max}(X,\ 0.3\ [\text{PS13}] + 0.7X)\end{aligned}$$

Please ask questions via **Piazza** or by email to **ganor@berkeley.edu**, and please don't reply to any **Piazza** questions during the exam.

We wish you all good luck!

UC Berkeley's Honor Code

"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

- I alone am taking this exam.
- I will not receive assistance from anyone while taking the exam nor will I provide assistance to anyone while the exam is still in progress.
- Other than with the instructor and GSI, I will not have any verbal, written, or electronic communication with anyone else while I am taking the exam or while others are taking the exam.

Problem 1 [15pts] – short answers

(a) Express $\mathrm{Re}[\sin(1+i)]$ in terms of $\sin 1$, $\cos 1$, e, but without any i.

(b) Which of these functions are analytic in $z = x + iy$?

(A) $e^{x+y}\sin(x-y) + e^{x+y}\cos(x-y)\,i,$ (B) $\arctan\left(\frac{y}{x}\right) - \frac{1}{2}\log(x^2+y^2)i,$

(C) $\left(x^3 - 3xy^2\right) + \left(3x^2y - y^3\right)i,$

Please explain your conclusions.

(c) In the range $-1 \le x \le 1$, the function e^x is expanded in Legendre polynomials $P_\ell(x)$ as follows:

$$e^x = \sum_{\ell=0}^{\infty} C_\ell P_\ell(x).$$

Find the coefficient C_1.

Problem 2 [15pts] – short answers

(a) We are given $f(x) = ax + b\sin x$ for some unknown constants a, b. Find functions $P(x)$ and $Q(x)$ such that $f(x)$ is a solution to the ordinary linear differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

for **all** values of a, b. Your answer for $P(x)$ and $Q(x)$ should not depend on the unknown a, b.

(b) Which of the following are **exact** differential equations?

(A) $0 = \left(\frac{2x}{y}\right)dx - \left(\frac{x^2}{y^2}\right)dy,$ (B) $0 = \left(\frac{y}{x^3}\right)dx + \left(\frac{1}{x^2}\right)dy,$

(C) $0 = \left(\frac{e^y}{x+y}\right)dx + e^y\left[\frac{1}{x+y} + \log(x+y)\right]dy$

(c) Write the standard notation for **one** solution to the ordinary differential equation

$$y'' + \left(\frac{1}{x}\right)y' + y = 0.$$

(You need to find a nonzero solution – the trivial $y = 0$ solution doesn't count for credit.)

Problem 3 [20pts] – medium length answer

Given that for any real number u:

$$\boxed{\int_{-\infty}^{\infty} e^{ixu} e^{-|u|} (\cos u)\, du = 2\left(\frac{2+x^2}{4+x^4}\right)}$$

(a) Calculate

$$\int_{-\infty}^{\infty} \left(\frac{2+x^2}{4+x^4}\right) e^{-ixu} dx.$$

Your answer should be expressed as a function of u only.

(b) Calculate

$$\int_{-\infty}^{\infty} \left(\frac{2+x^2}{4+x^4}\right)^2 dx.$$

Hint: You can do these problems with a contour integral, which is OK, but time consuming. There is another way that is much shorter, used the boxed equation, and does not involve contour integrals. Either way, you'll get full credit for a correct solution with complete explanation.

Problem 4 [20pts] – medium length answer

(a) Solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{2x^3 y}{x^4 + y^4}$$

with the condition that $y = 1$ at $x = \sqrt{2}$. You can leave the solution in implicit form $F(x, y) = 0$.

Hint: You might need the integral

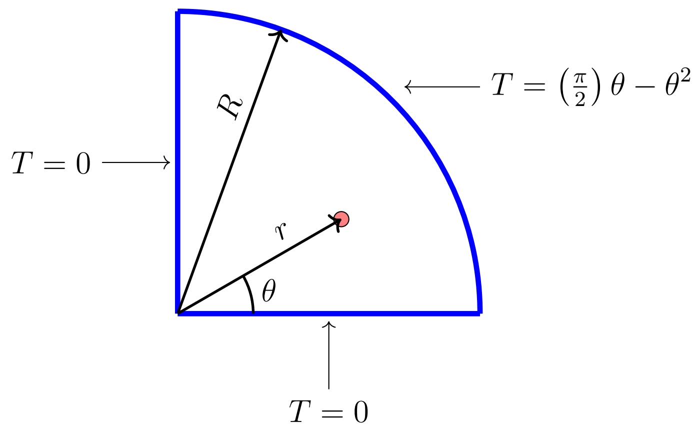
$$\int \left(\frac{Au^4 + B}{Cu^5 - Du}\right) du = -\left(\frac{B}{D}\right) \log u + \left(\frac{AD + BC}{4CD}\right) \log\left(D - Cu^4\right) \qquad \text{for constant } A, B, C, D.$$

(b) Find the general solution to the ordinary differential equation

$$y' + (\tan x)\, y + \frac{2\cos x}{x^2} = 0.$$

Note: if you use $\log(\cdots)$, you may add an assumption that the argument $(\cdots)$ is positive. [You don't have to consider separately the case $(\cdots) < 0$, which technically would be required for completeness.]

Problem 5 [20pts] – longer answer



Solve the 2d Laplace equation inside the quadrant shown in the picture, with boundary conditions given below. In polar coordinates, the quadrant is given by

$$0 \le \theta \le \frac{\pi}{2} \qquad \text{and} \qquad 0 \le r \le R.$$

Laplace's equation for $T(r, \theta)$ takes the form

$$\boxed{0 = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\left(\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \theta^2}\right)}$$

The boundary conditions are:

$$\boxed{T(r,0) = T(r,\frac{\pi}{2}) = 0} \qquad \text{and} \qquad \boxed{T(R,\theta) = \left(\frac{\pi}{2}\right)\theta - \theta^2} \qquad \text{for } 0 \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le R.$$

Your final solution can be in the form

$$T(r,\theta) = \sum_{n=1}^{\infty}(\cdots).$$

You can (but don't have to) follow the steps below:

(a) Start by identifying solutions to $\nabla^2 T = 0$ of the form $T(r,\theta) = f(r)g(\theta)$ that satisfy $T(r,0) = T(r,\frac{\pi}{2}) = 0$ (but not necessarily the boundary condition at $r = R$).

(b) Then look for a solution which is a linear combination of the various solutions of part (a). You can label the solutions by $n = 1, 2, \ldots$ and write the linear combination in the form $\sum_n C_n f_n(r) g_n(\theta)$. Find the constants C_n.

You might need the integrals:

$$\int \sin(au)du = -\frac{1}{a}\cos(au)\,, \qquad \int u\sin(au)du = \left(\frac{1}{a^2}\right)\sin(au) - \left(\frac{u}{a}\right)\cos(au)\,,$$

$$\int u^2\sin(au)du = \left(\frac{2u}{a^2}\right)\sin(au) + \left(\frac{a^2u^2-2}{a^3}\right)\cos(au)\,.$$

Problem 6 [20pts] – longer answer

(a) Find the first 5 terms $(a_0, a_1, a_2, a_3, a_4)$ in a series solution $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for the 3^{rd} order ordinary differential equation:

$$x^2 f'''(x) + x f'(x) + f(x) = 0$$

with initial conditions

$$f(0) = f'(0) = 0, \qquad f''(0) = 2.$$

(b) The solution of part(a) can be expressed in terms of the Bessel function of order 1 as

$$f(x) = A x^{\alpha} J_1(B x^{\beta}),$$

for some constant coefficients A, B, α, β. Find these four constants.

Hint: one way to solve this problem is to expand $f(x)$ in a Taylor series, and use the Taylor series

$$J_1(u) = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{2^{2n+1} n!(n+1)!}$$

to compare.