Physics 89 Midterm Exam
Thursday, 10/15/2020, 12:40-2:10pm (PT)

Name or SID: ____________

Instructions

1. Please submit the exam using **Gradescope**.
2. Please solve all 3 problems below.
3. The maximal score is 102.

The Lecture zoom link will be open during the exam, and you can ask questions via the **chat**, or by email.

Good luck!

UC Berkeley's Honor Code

"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

- I alone am taking this exam.
- I will not receive assistance from anyone while taking the exam nor will I provide assistance to anyone while the exam is still in progress.
- Other than with the instructor and GSI, I will not have any verbal, written, or electronic communication with anyone else while I am taking the exam or while others are taking the exam.

Problem 1 [34pts] – short answers

(a) Define the two functions $f(x)$ and $g(x)$ by

$$f(x) = \frac{1}{x^2+6x+5}, \qquad g(x) = \frac{1}{x^2+4x+5}.$$

The two Taylor series

$$\begin{aligned} f(x) &= \frac{1}{x^2+6x+5} = \tfrac{1}{5} - \tfrac{6}{25}x + \tfrac{31}{125}x^3 + \cdots \\ g(x) &= \frac{1}{x^2+4x+5} = \tfrac{1}{5} - \tfrac{4}{25}x + \tfrac{11}{125}x^3 + \cdots \end{aligned}$$

turn out to have different segments of convergence. The series for $g(x)$ converges for $|x| < \sqrt{5}$ and doesn't converge for $|x| > \sqrt{5}$, while the series for $f(x)$ converges for $|x| < 1$ and doesn't converge for $|x| > 1$.
Can you explain this fact using complex numbers?
[It is not important for this problem what happens at $x = \pm 1$ for $f(x)$ and $x = \pm\sqrt{5}$ for $g(x)$.]

(b) Recall our matrix notation for a linear system of equations in 3 variables X, Y, Z:

$$\left.\begin{aligned} aX + bY + cZ &= p \\ dX + eY + fZ &= q \\ gX + hY + lZ &= s \end{aligned}\right\} \Longrightarrow \qquad \mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & l \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a & b & c & p \\ d & e & f & q \\ g & h & l & s \end{pmatrix}.$$

Suppose after some **row-operations** we bring **A** to the form

$$\xrightarrow{\text{Row operations}} \begin{pmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 0 & T \end{pmatrix}.$$

For which value(s) of T, if any, will there be a unique solution for X, Y, Z?
For which value(s) of T, if any, will there be no solutions?
For which value(s) of T, if any, will there be more than one solution?

Problem 2 [34pts]

For each of the two differential equations,

$$y'(t) - y(t) = \cos(2t), \qquad y'(t) - y(t) = \sin(2t),$$

find a solution $y(t)$ using complex number methods.

Note 1: $y(t)$ is an unknown function of t that you need to find, and $y'(t)$ is its (unknown) derivative.
Note 2: Your final results should be *real* and should not contain any i, or $\text{Re}(\cdots)$ or $\text{Im}(\cdots)$.

Problem 3 [34pts]

Given the matrix below

$$\mathbf{M} = \begin{pmatrix} 1 & -2 \\ -4 & 3 \end{pmatrix},$$

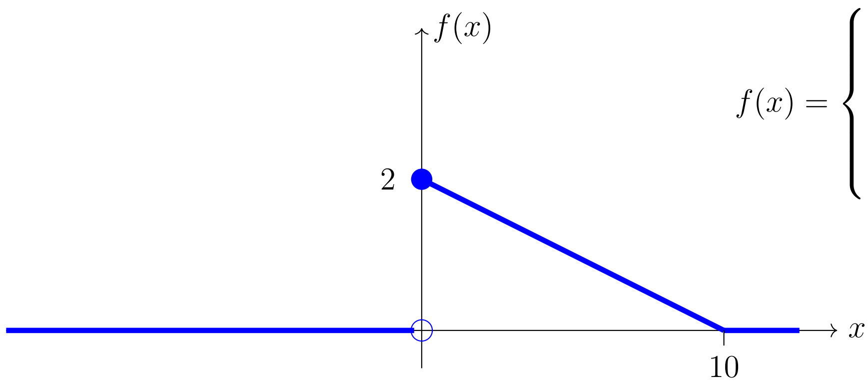
solve the following problems:

(a) Find all the eigenvalues of $\mathbf{M}$.

(b) For each eigenvalue from part (a), find a corresponding eigenvector.

(c) Find a matrix $\mathbf{C}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{MC} = \mathbf{CD}$.

(d) Apply the function whose graph is depicted below to the matrix $\mathbf{M}$.
In other words, what is $f(\mathbf{M})$?



$$f(x) = \begin{cases} 0 & \text{for } x < 0; \\ 2 - \frac{1}{5}x & \text{for } 0 \le x \le 10; \\ 0 & \text{for } x > 10; \end{cases}$$