Physics 7B Fall 2020
Lecture 1 Midterm 2 Solutions

Problem 1

1. (a) Labeling the charges as a,b,c from left to right,

$$\begin{aligned}
\mathbf{E} &= \mathbf{E}_a + \mathbf{E}_b + \mathbf{E}_c \quad \text{(1 point)} \\
&= (E_{a,y} + E_{c,y} + E_b)\hat{z} \quad \text{(3 points)} \\
&= ((E_a + E_c)\sin(\theta) + E_c)\hat{z} \quad \text{(2 points)} \\
&= (E_a + E_c)\frac{z}{\sqrt{z^2+l^2}} + E_c)\hat{z} \quad \text{(2 points)} \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{2Qz}{(z^2+l^2)^{3/2}} - \frac{2Q}{z^2}\right]\hat{z} \quad \text{(3 points)} \\
&= \frac{-Q}{4\pi\epsilon_0}\left[\frac{1}{z^2} - \frac{z}{(z^2+l^2)^{3/2}}\right]\hat{z} \quad \text{(1 point)}
\end{aligned}$$

(b) The long distance behavior is obtained by assuming

$$z >> l \rightarrow \frac{l}{z} << 1 \quad \text{(1 point).}$$

This requires a Taylor expansions. Rearranging the expression for $\mathbf{E}$ above,

$$\mathbf{E} = \frac{-Q}{4\pi\epsilon_0}\frac{1}{z^2}\left[1 - \left(1 + \left(\frac{l}{z}\right)^2\right)^{-3/2}\right]\hat{z} \quad \text{(1 point)}$$

Nothing the expansion

$$\left(1 + \left(\frac{l}{z}\right)^2\right)^{-3/2} = 1 - \frac{3}{2}\left(\frac{l}{z}\right)^2 + ... \quad \text{(1 point)}$$

We obtain

$$\mathbf{E} \approx \frac{-3Ql^2}{4\pi\epsilon_0 z^4}\hat{z} \quad \text{(1 point)}$$

(c) To compare with the electric dipole field, we first note that the field of a dipole falls off like $E \sim 1/r^3$ (1 point). This is evidently different than the result obtained for the quadrupole, which falls off like $E \sim 1/r^4$ (1 point). The physical insight is that the quadrupole is like two dipoles in opposite directions, and so just as the dipole field is smaller than individual point charges due to “cancellation” of charges, the quadrupole field is smaller than the individual dipole fields (2 points).

Other accepted physical explanations are that for the quadrupole, both the monopole and dipole moments are zero, and so the first nonzero component to the field in a multipole expansion will look like $E_{\mathrm{quad}} \sim I^2/r^6 \sim 1/r^4$, where I is the quadrupole moment, $I \sim r$.

Problem 2 - Solution and Rubric

1 Solution

1.1 Method 1: Direct Calculation

There is an azimuthal symmetry here, so we need only calculate the electric field along some line on the curved surface, and symmetry gives us the rest. Let $z = 0$ be the center, where the charge Q is located. A point z along the length and directly above will have a separation:

$$\vec{r} = R_0\hat{\rho} + z\hat{\mathbf{z}} \tag{1}$$

where $\hat{\rho}$ is the radial unit vector in cylindrical coordinates. Coulomb's law gives:

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0}\frac{R_0\hat{\rho} + z\hat{\mathbf{z}}}{(R_0^2 + z^2)^{3/2}} \tag{2}$$

This is the same around any ring of constant z, so we can write:

$$\Phi = 2\pi R_0 \int_{-R_0}^{R_0} \vec{E}\cdot\hat{\rho}dz \tag{3}$$

$$= \frac{QR_0^2}{2\epsilon_0}\underbrace{\int_{-R_0}^{R_0}\frac{dz}{(R_0^2 + z^2)^{3/2}}}_{=\sqrt{2}/R_0^2} \tag{4}$$

$$\implies \boxed{\Phi = \frac{\sqrt{2}Q}{2\epsilon_0} = \frac{Q}{\sqrt{2}\epsilon_0}} \tag{5}$$

1.2 Method 2: Subtract the ends

Gauss's law tells us that the total flux (curved side plus the ends) should be:

$$\Phi_{\text{total}} = \frac{Q}{\epsilon_0} \tag{6}$$

So, we could calculate the flux through one of the ends, double it (by symmetry), and subtract from Q/ϵ_0 to get the flux through the curved side. If we break the ends into infinitesimal rings, each ring will have a constant separation distance. Moreover, we need only the normal component for the flux calculation. Consider the infinitesimal ring of radius $0 \leq r \leq R_0$. It has normal separation vector, and separation distance, given by:

$$\vec{r}_\perp = R_0\hat{\mathbf{z}} \tag{7}$$

$$|\vec{r}| = \sqrt{R_0^2 + r^2} \tag{8}$$

We can then calculate:

$$\Phi_{\text{end}} = \frac{Q}{4\pi\epsilon_0}\int_0^{R_0}\frac{R_0 2\pi r dr}{(R_0^2 + r^2)^{3/2}} \tag{9}$$

$$= \frac{Q}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{2}}\right] \tag{10}$$

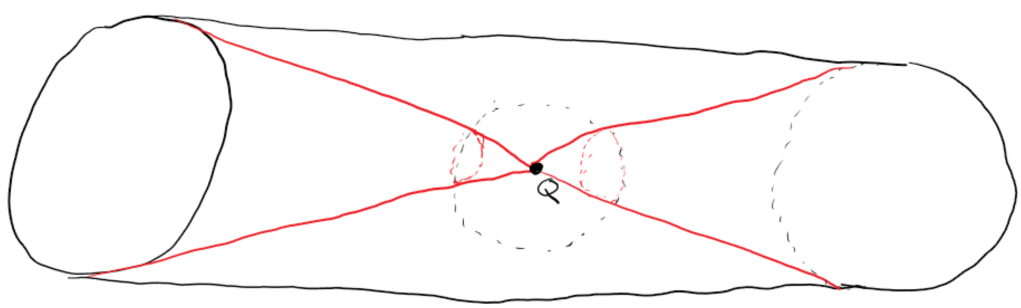
We then have:

$$\Phi_{\text{sides}} = \frac{Q}{\epsilon_0} - 2\frac{Q}{2\epsilon_0}\left[1 - \frac{1}{\sqrt{2}}\right] \tag{11}$$

$$\implies \boxed{\Phi_{\text{sides}} = \frac{Q}{\sqrt{2}\epsilon_0}} \tag{12}$$

1.3 Method 3: Solid angle / Spherification

Imagine a small sphere drawn centered around the charge, inside the cylinder in question. All of the flux that passes through this sphere will pass through the cylinder, and vice-versa. The flux that passes through the ends of the cylinder can be traced back to some portion of the sphere, corresponding to the same solid angle as shown below:



The advantage of this method is that it is much easier to calculate the flux through a portion of the sphere, since the electric field of a point charge is constant around a spherical surface. The total flux is given by Gauss's law as $\Phi_{\text{total}} = Q/\epsilon_0$. Since the radius and half-length of the cylinder are both R_0, we get a cone-angle of $\pi/4$. We can calculate the portion of the area of the sphere this takes up like so:

$$A_{\text{cone-bounded}} = 2\pi R^2 \int_0^{\pi/4} \sin\theta d\theta = 2\pi R^2 \left[1 - \frac{1}{\sqrt{2}}\right] \tag{13}$$

for some R. The total area is $4\pi R^2$, so we take:

$$\Phi_{\text{side}} = \left[1 - \frac{2A_{\text{cone-bounded}}}{4\pi R^2}\right] \frac{Q}{\epsilon_0} \tag{14}$$

$$\implies \boxed{\Phi_{\text{side}} = \frac{Q}{\sqrt{2}\epsilon_0}} \tag{15}$$

2 Rubric

The total possible points for this problem is 20. Partial credit is awarded according to the table below:

Scoring	Reason
+5	Notes the relevance of Gauss's Law in any way
+15	Correct result or only trivial errors (i.e. dropped a factor of 2, etc.)
+10	Almost correct, but with a non-trivial error (i.e. wrong units, wrong integrals set up, etc.)
+10	Correct result, but missing some justifications (i.e. wrote an intermediate flux without the way it is calculated, etc.)
+10	Correct result, but un-reduced (i.e. left in integral form, etc.)
+5	Significant mistakes (i.e. saying electric field is constant where it isn't, etc.)
+0	Incorrect methods and result (i.e. calculating wrong thing, incorrect setup, etc.)
+0	No relevant work submitted (i.e. no submission or left blank)

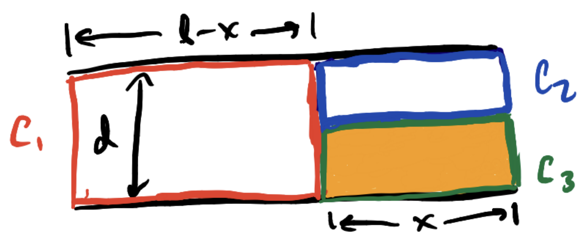
For example, a submission that just states that one could use Gauss's law and subtract the ends, without actually calculating anything, would receive a 5/20. A submission that did everything right except for, say, forgetting a factor of 2 in the integral of something, would receive a 20/20. A submission that set up all the right integrals but did not evaluate them would get a 15/20, and so on.

Problem 3 Solution and Rubric

Solution

1. (a) To begin with, we decompose the capacitor as the parallel or series sum of three sub-capacitors as shown in the figure:

Figure 1: Decomposition of capacitor



Capacitors C_1 and C_2 are parallel plate capacitors with no dielectric media, C_1 having plate area $\ell(\ell - x)$ and plate separation d and C_2 having plate area ℓx and plate separation $d/2$. Capacitor C_3 is a parallel plate capacitor with dielectric media K, plate area ℓx, and plate separation $d/2$. With these specifications, then

$$\begin{aligned} C_1 &= \frac{\epsilon_0 \ell(\ell - x)}{d}, \\ C_2 &= \frac{2\epsilon_0 \ell x}{d}, \\ C_3 &= \frac{2K\epsilon_0 \ell x}{d}. \end{aligned}$$

Since C_2 and C_3 are plate to plate, they are in series, so we can combine them into an equivalent capacitor $C_{2,3}$ with capacitance

$$C_{2,3} = \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} = \left[\frac{d}{2\epsilon_0 \ell x}\left(1 + \frac{1}{K}\right)\right]^{-1} = \frac{2K}{K+1}\frac{\epsilon_0 \ell x}{d}.$$

Now, we note that the positive plates of C_1 and $C_{2,3}$ are part of the same conductor, and the negative plates of C_1 and $C_{2,3}$ are part of the same conductor. This means that the positive plates share a potential, and the negative plates share a separate potential, leading to the conclusion that the potential difference between the plates is the same for both. Thus, C_1 and $C_{2,3}$ are in parallel, and we can add them as such to get the total capacitance C of our system:

$$\begin{aligned} C = C_1 + C_{2,3} &= \frac{\epsilon_0 \ell(\ell - x)}{d} + \frac{2K}{K+1}\frac{\epsilon_0 \ell x}{d} \\ &= \frac{\epsilon_0 \ell^2}{d}\left(1 - \frac{x}{\ell} + \frac{2K}{K+1}\frac{x}{\ell}\right) = \frac{\epsilon_0 \ell^2}{d}\left(1 + \frac{K-1}{K+1}\frac{x}{\ell}\right). \end{aligned}$$

(b) We can use the formula $U_{\text{cap}} = \frac{1}{2}CV^2$ here:

$$U_{\text{cap}} = \frac{1}{2}CV^2 = \frac{1}{2}\left[\frac{\epsilon_0 \ell^2}{d}\left(1 + \frac{K-1}{K+1}\frac{x}{\ell}\right)\right]V_0^2 = \frac{\epsilon_0 \ell^2 V_0^2}{2d}\left(1 + \frac{K-1}{K+1}\frac{x}{\ell}\right)$$

(c) Our strategy here will be to first compute the potential energy of the system as a function of x, and then differentiate with respect to x to get the force. This total potential energy can be considered as a sum of the stored potential of the capacitor and the potential energy of whatever is keeping the electric potential between the capacitors equal to V_0. We already have $U_{\text{cap}}(x)$, but the other potential $U_{\text{batt}}(x)$ we have yet to calculate. To do so, we note that, as the capacitance changes with x, the potential between the plates would change if the charge weren't forced to move from one plate to the other. More specifically, in order to maintain a constant potential V_0 given a change in capacitance ΔC, we need a change in charge $\Delta Q = V_0 \Delta C$. This can be realized by moving a charge $V_0 \Delta C$ from the negative plate to the positive plate, but this changes the energy of these charges by a shift of $\Delta U_Q = \Delta Q V_0 = V_0^2 \Delta C$ since these charges traverse a potential difference of V_0 in going from the negative plate to the positive plate. In order to compensate for this energy change, the potential U_{batt} must then change by $-V_0^2 \Delta C$. So, if the battery begins with a potential energy U_0, then in the process of inserting the dielectric the energy changes to

$$\begin{aligned} U_{\text{batt}}(x) &= U_0 + \Delta U_{\text{batt}}(x) = U_0 - V_0^2 \Delta C(x) = U_0 - V_0^2(C(x) - C(0)) \\ &= U_0 - V_0^2 \times \frac{\epsilon_0 \ell^2}{d}\frac{K-1}{K+1}\frac{x}{\ell} = U_0 - \frac{K-1}{K+1}\frac{\epsilon_0 \ell^2 V_0^2}{d}\frac{x}{\ell}. \end{aligned}$$

We are now ready to write the total potential:

$$U(x) = U_{\text{cap}}(x) + U_{\text{batt}}(x) = U_0 + \frac{\epsilon_0 \ell^2 V_0^2}{2d}\left(1 - \frac{K-1}{K+1}\frac{x}{\ell}\right).$$

Then, the force in the x direction is given as

$$F_x = -\frac{d}{dx}U(x) = +\frac{K-1}{K+1}\frac{\epsilon_0 \ell V_0^2}{2d}.$$

Since x increases to the left and F_x is positive, the force points to the left.

Rubric

(a) Total: 10 points

i. Split the capacitor into subcapacitors (1 points)
ii. Choose a correct decomposition (1 point)
iii. Correctly state/use $C = \frac{K\epsilon_0 A}{d}$ (2 points)
iv. Correctly find capacitances of subcapacitors (1 point)
v. Correctly state/use the series capacitance addition law (2 points)
vi. Correctly state/use the parallel capacitance addition law (2 points)
vii. Correctly find the total capacitance in terms of given decomposition (1 point)

(b) Total: 4 points

i. Correctly state/use some form of $U = \frac{1}{2}CV^2$ for a capacitor (3 points)
ii. Give a correct answer in terms of previous work (1 point)

(c) Total: 6 points

i. Correctly state/use some form of $F_x = -\frac{d}{dx}U(x)$ (2 points)

ii. Include the capacitor energy from (b) in $U(x)$ (1 point)
iii. Include the "battery" energy in $U(x)$ (1 point)
iv. Get the correct magnitude in terms of previous answers (1 point)
v. Get the correct direction in terms of previous answers (1 point)

Problem 4

(a) Power $P = VI$, so $I = P/V$. Thus, $I_A = P_A/V_A = 50\,\mathrm{W}/130\,\mathrm{V} = 0.38\,\mathrm{A}$, and $I_B = P_B/V_B = 50\,\mathrm{W}/11\,\mathrm{V} = 4.55\,\mathrm{A}$.

(b) By Ohm's law, $R = V/I$. So $R_A = V_A/I_A = 130\,\mathrm{V}/(5/13\,\mathrm{A}) = 338\,\Omega$, and $R_B = V_B/I_B = 11\,\mathrm{V}/(50/11\,\mathrm{A}) = 2.42\,\Omega$.

(c) Energy is $E = Pt$. In one hour, $t = 3600\,\mathrm{s}$. The power is $50\,\mathrm{W}$ for both bulbs, so the energy $E_A = E_B = (50\,\mathrm{W})(3600\,\mathrm{s}) = 1.8 \times 10^5\,\mathrm{J}$.

(d) Bulb B requires a larger current, and so should have larger diameter connecting wires to avoid overheating the connecting wires.

Rubric:

(a) Write down $P = IV$, +3 pt

Isolate I in terms of P and V correctly, +1 pt.

Correct numerical result, +1 pt.

Correct numerical result without isolating I, +2 pt.

(b) Write down Ohm's law, +3 pt.

Isolate R in terms of V and I (or V and P), +1 pt.

Correct numerical result, +1 pt.

Correct numerical result without isolating R, +2 pt.

(c) Write down energy $E = Pt$, +3 pt.

Time in correct units, +1 pt.

Correct numerical result, +1 pt.

(d) Correctly identify bulb B requires larger diameter, +2 pt.

Correct explanation (power dissipation), +3 pt.

Reasonable explanation (mention larger current), +2 pt.

Some explanation, e.g. about wire *inside* the bulb, +1 pt.

Problem 5 Solution

(a) We will approach this using the hint, continuing to add charge to the conducting sphere. Suppose the sphere is already charged with charge q (uniformly distributed on the surface): the energy required to add additional charge dq to the surface of the sphere (from infinity) is given by

$$dU = dq\left(V\left(r = r_0\right) - V\left(r = \infty\right)\right),$$

where $V(r)$ is the electric potential. By Gauss' Law with a spherical Gaussian surface,

$$\mathbf{E}(r) = \hat{r}\frac{q}{4\pi\epsilon_0 r^2}$$

for $r > r_0$. The electric potential is thus given by

$$V\left(r = r_0\right) - V\left(r = \infty\right) = -\int_{\infty}^{r_0} \mathbf{E}(r) \cdot \hat{r} dr = \frac{q}{4\pi\epsilon_0 r_0},$$

which is the same as that of a point charge. We therefore write

$$dU = \frac{qdq}{4\pi\epsilon_0 r_0}$$

and integrate to get the total electrostatic energy,

$$U = \int dU = \int_0^Q \frac{qdq}{4\pi\epsilon_0 r_0} dr = \boxed{\frac{Q^2}{8\pi\epsilon_0 r_0}.}$$

(b) We will construct this charge configuration by sequentially adding charged spherical shells, each with the same constant charge density as the final charged sphere,

$$\rho = \frac{Q}{\frac{4}{3}\pi {r_0}^3}.$$

Suppose we have added spherical shells up to radius r and then add another shell of thickness dr. The charge of this spherical shell is given by the product of the charge density and the infinitesimal volume, namely

$$dq = \rho 4\pi r^2 dr.$$

The energy required to add this spherical shell (to radius r) is again given by

$$dU = dq\left(V(r) - V(\infty)\right).$$

$V(r) - V(\infty)$ is again the same for this configuration as for a point charge, so we can write

$$V(r) - V(\infty) = \frac{q}{4\pi\epsilon_0 r},$$

where q is the total charge added so far:

$$q = Q\frac{r^3}{{r_0}^3}.$$

Combining,

$$dU = dq\left(V(r) - V(\infty)\right) = \frac{Q}{\frac{4}{3}\pi r_0^3} 4\pi r^2 dr \frac{Q\frac{r^3}{{r_0}^3}}{4\pi\epsilon_0 r} = \frac{3Q^2 r^4}{4\pi\epsilon_0 {r_0}^6} dr.$$

Therefore, the total electrostatic energy is given by

$$U = \int dU = \int_0^{r_0} \frac{3Q^2 r^4}{4\pi\epsilon_0 {r_0}^6} dr = \boxed{\frac{3Q^2}{20\pi\epsilon_0 r_0}.}$$

Bonus Question

A proton is made up of 2 Up quarks and 1 Down quark.

a) What is the quark composition of the neutron? (3 points)

ANSWER: Neutrons are made of 1 Up quark and 2 down quarks.

b) How are the quarks held together? (2 points)

ANSWER: Quarks are held together by the strong nuclear force.

Rubric

a) No partial credit. Answer must contain 1 Up and 2 Down for credit.

b) 1 point: nuclear force. 1 point: strong force.