EECS 16A Designing Information Devices and Systems I
Spring 2017 Babak Ayazifar, Vladimir Stojanovic **Midterm 1**

Exam location: 145 Dwinelle, last SID# 2

PRINT your student ID: ________________________________________

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-_____________

PRINT your discussion section and GSI (the one you attend): ________________________________________

Name and SID of the person to your left: ________________________________________

Name and SID of the person to your right: ________________________________________

Name and SID of the person in front of you: ________________________________________

Name and SID of the person behind you: ________________________________________

**1. What other courses are you taking this term? (1 point)**

**2. What activity do you really enjoy? Describe how it makes you feel. (1 point)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

PRINT your name and student ID: ____________________________________________

**3. Mechanical Basis (8 points)**

(a) (3 points) Let vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3 \in \mathbb{R}^4$ :

$$\vec{v}_1 = \begin{bmatrix}1\\2\\3\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}2\\0\\3\\0\end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for the vector space $\mathbb{R}^4$? Justify your answer.

**Solutions:**

No, there are only 3 vectors, but in order to span $\mathbb{R}^4$, you need at least 4 vectors so that every vector in the set can be formed by a linear combination of the basis vectors.

(b) (5 points) Let $\vec{x} = \begin{bmatrix}3\\5\\4\end{bmatrix}$. Given a new set of vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix}1\\3\\0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}2\\7\\1\end{bmatrix}, \vec{v}_4 = \begin{bmatrix}2\\0\\2\end{bmatrix}, \vec{v}_5 = \begin{bmatrix}1\\6\\3\end{bmatrix}$$

Can the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be a basis for $\mathbb{R}^3$? If so, express $\vec{x}$ as a linear combination of the basis vectors. Otherwise, choose a new basis using $\vec{v}_1$, $\vec{v}_2$ and any number of additional vectors in the set, then express $\vec{x}$ as a linear combination of the newly constructed basis vectors.

**Solutions:**

No, not a basis since there are 5 vectors for a 3 dimensional space, and a basis is defined as having the minimum number of vectors to span the vector space. Since we are told to use $\vec{v}_1$ and $\vec{v}_2$ in the basis, we only need to choose the appropriate 3rd vector: a vector that is linearly independent with $\vec{v}_1$ and $\vec{v}_2$. By inspection, or using any other methods, we see that

$$\vec{v}_3 = 2\vec{v}_1 + \vec{v}_2$$

$$\vec{v}_5 = \vec{v}_1 + 3\vec{v}_2$$

Therefore, $\vec{v}_4$ is the only vector that is linearly independent with $\vec{v}_1$ and $\vec{v}_2$. So $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ form the basis.

We can use Gaussian elimination to determine the coefficients:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 3 & 1 & 0 & 5 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -6 & -4 \\ 0 & 1 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -6 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore, we get the linear combination

$$\vec{x} = 1\vec{v}_1 + 2\vec{v}_2 + 1\vec{v}_4$$

PRINT your name and student ID: ________________________________________________

## 4. Eigenvectors (10 points)

(a) (5 points) Find the eigenvectors and associated eigenvalues of $\mathbf{M}$ in terms of $a$ and $b$.

$$\mathbf{M} = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$$

**Solutions:** The characteristic polynomial of $\mathbf{M}$ is $(\lambda - 1)(\lambda - b) = 0$, so we have $\lambda = 1,\ b$. Then we solve for eigenvectors by finding the null space of $\mathbf{M} - \lambda \mathbf{I}$ for each $\lambda$.

$$\mathbf{M}\vec{x} = \lambda \vec{x}$$

$$\mathbf{M}\vec{x} - \lambda \mathbf{I}\vec{x} = \vec{0}$$

For $\lambda = 1$:

$$\begin{bmatrix} 0 & a \\ 0 & b-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Where we see the eigenvector by inspection because the first column is zero.

For $\lambda = b$:

$$\begin{bmatrix} 1-b & a \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

$$(1-b)x_1 + a x_2 = 0$$

$$(b-1)x_1 = a x_2$$

$$\vec{v} = \begin{bmatrix} a \\ b-1 \end{bmatrix}$$

(b) (5 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & -2 \\ 0 & 0.5 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Evaluate $\mathbf{A}^{203}\mathbf{B}^{199}\vec{v}$.

**Solutions:** Observe that $\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with $\lambda = 2$ and also an eigenvector of $\mathbf{B}$ with $\lambda = 0.5$. Viewing the desired product as a sequence of multiplications:

$$\mathbf{A}(\mathbf{A}(\ldots \mathbf{A}(\mathbf{B}(\mathbf{B}(\ldots \mathbf{B}(\mathbf{B}\vec{v}))\ldots))$$

Every multiplication with $\mathbf{B}$ will scale the result by 0.5 and then every multiplication with $\mathbf{A}$ will scale the result by 2. Therefore, we can simply write

$$\mathbf{A}^{203}\mathbf{B}^{199}\vec{v} = 2^{203} 0.5^{199} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= 2^4 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 64 \\ 16 \end{bmatrix}$$

Common mistakes:

- Arithmetic mistakes
- Not using the eigenvalue/eigenvector definition
- Trying to discover a relationship for high powers of the matrices $\mathbf{A}$ or $\mathbf{B}$.

**5. Eigenvalue Proof (10 points)**

For two square matrices $\mathbf{A}$ and $\mathbf{B}$, show that $\mathbf{AB}$ has the same eigenvalues as $\mathbf{BA}$.

*Hint: Show that if $\mathbf{AB}$ or $\mathbf{BA}$ has an eigenvalue $\lambda$, then the other one has the same eigenvalue.*

**Solutions:**

$$\mathbf{AB}\vec{x} = \lambda\vec{x}$$
$$\mathbf{BAB}\vec{x} = \mathbf{B}\lambda\vec{x}$$
$$\mathbf{BA}(\mathbf{B}\vec{x}) = \lambda(\mathbf{B}\vec{x})$$

Let $\vec{y} = \mathbf{B}\vec{x}$. Then $\mathbf{BA}\vec{y} = \lambda\vec{y}$, so $\mathbf{BA}$ has the same eigenvalues $\lambda$ as $\mathbf{AB}$ with the corresponding eigenvector(s) $\vec{y}$.

**6. Wall Shadows (15 points)**

Oh no, someone decided to build a wall between you and your favorite neighbor! They built a huge complicated 3D structure that you cannot fully see. However, you can observe the shadow cast by this structure.

(a) (5 points) Let's first try to model what the shadow of a 3D object might be. Create a transformation that flattens all components of a vector (vectors in 3D space) onto the $xy$-plane. That is, find a matrix $\mathbf{A}$ such that :

$$\mathbf{A}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

**Solutions:**

Dimension matching says $\mathbf{A}$ must be a $3 \times 3$ matrix. Its values can be found via brute force matrix multiplication or by modifying the identity matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) (5 points) Suppose you wanted to observe the height of the wall. Could you observe it just by seeing the shadow, assuming the shadow is created by the transformation from the matrix above?

**Solutions:** Any argument based on linear dependence and the lack of the existence of an inverse is acceptable.

(c) (5 points) One of your friends comes along and wonders, if you apply the matrix again, whether you will get any new information. Realizing your matrix is really just a projection, you argue that projecting twice will give you no new information. We will show this generally for any two vectors. To prove this to your friend, show the following:

For any vectors $\vec{u}$ and $\vec{v}$, if $\vec{x} = proj_{\vec{u}}\vec{v}$ and $\vec{y} = proj_{\vec{u}}\vec{x}$, then $\vec{x} = \vec{y}$.
Recall that the projection of a vector $\vec{v}$ onto $\vec{u}$, $proj_{\vec{u}}\vec{v} = \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u}$.

**Solutions:**

The geometric intuition here is that if you apply a projection once, the output will be in the subspace you are projecting on. If you apply the same projection to the resulting vector, it is already in the subspace, so the vector will not change.

Algebraically, we use the inner product definition of projection:

$$\begin{aligned}
\vec{x} = proj_{\vec{u}}\vec{v} &= \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u} \\
\vec{y} = proj_{\vec{u}}\vec{x} &= \frac{\langle \vec{x}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u} \\
&= \frac{\langle \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u} \\
&= \frac{\frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\langle \vec{u}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u} \\
&= \frac{\langle \vec{v}, \vec{u} \rangle}{||\vec{u}||^2}\vec{u} \\
&= \vec{x}
\end{aligned}$$

**7. Structured Illumination (20 points)**

In the lab, you acquired images using a single pixel imager and a projector. You did this by successively creating different illumination patterns (masks) and recording the total intensity at the single pixel detector. In this problem, consider a 3×3 illumination grid, where we will use structured light patterns (not just single-pixel masks) to acquire image information.

The 9 pixels, represented by $x_i$, have values that are either 0 (no light reflected by the object at that pixel location) or 1 (light is completely reflected by object at that pixel location). For example, if you use your sensor to look at the Campanile, pixels $x_2$, $x_5$, and $x_8$ would reflect light from an illumination source, or for Cory Hall, the bottom two rows would reflect light, as shown in Fig. 1

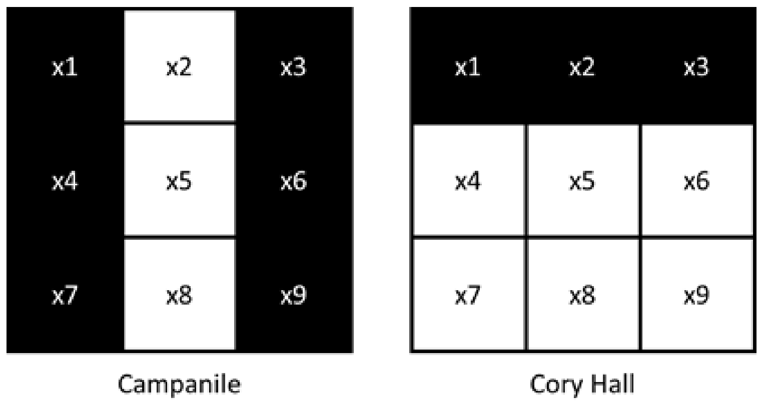



Figure 1: Campus objects represented with 9 pixels

For this problem, we will try to use just two mask patterns to recognize various objects around campus using our sensor.

(a) (2 points) For this part, consider the mask patterns shown in Fig. 2.

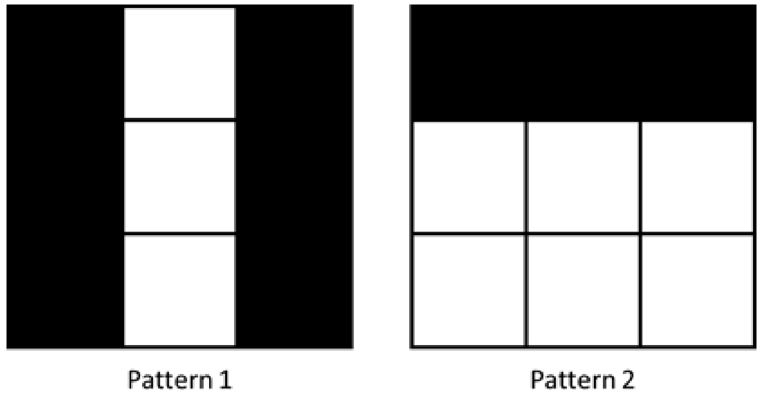



Figure 2: Two scanning patterns

The value detected at our light sensor for a particular mask pattern is the dot-product of the mask pattern with the 9-pixel representation of the object. Using the two mask patterns in Fig. 2, write down a matrix **K** to transform some image represented by a 9-element vector, $\vec{x}$, into a 2-element vector, $\vec{y}$ ($\mathbf{K}\vec{x} = \vec{y}$).

**Solutions:** The first row of **K** will correspond to the first mask, and the second row of **K** will correspond to the second mask, so

$$\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) (2 points) The objects we will image first are 9-pixel representations of the Campanile and Cory Hall, shown in Fig. 1. Write down two vectors, $\vec{x}_{campanile}$ and $\vec{x}_{cory}$ that represent the image data shown in Fig. 1, using 0 for dark pixels and 1 for light pixels.

**Solutions:** These can be read directly from the images as labeled and follow the same conventions as in the lab: $\vec{x}_{campanile} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ ; $\vec{x}_{cory} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(c) (2 points) Use the $\mathbf{K}$ matrix from part a) to transform $\vec{x}_{campanile}$ and $\vec{x}_{cory}$ into 2-element vectors, $\vec{y}_{campanile}$ and $\vec{y}_{cory}$. (That is, compute $\mathbf{K}\vec{x}_{campanile} = \vec{y}_{campanile}$ and $\mathbf{K}\vec{x}_{cory} = \vec{y}_{cory}$.)

**Solutions:** These can easily be computed by inspection:

$$\vec{y}_{campanile} = \mathbf{K}\vec{x}_{campanile} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\vec{y}_{cory} = \mathbf{K}\vec{x}_{cory} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

(d) (4 points) Based on your results in part c), how would you use the elements of your output vector $\vec{y}$ to distinguish between the Campanile and Cory Hall?

**Solutions:** Answers may vary, but one could use the relative size of the elements in the vector: if $y_1 > y_2$, $\vec{x}$ is the Campanile, otherwise $\vec{x}$ is Cory Hall. Answers that explain correct differences between the Campanile and Cory output vectors are valid.

(e) (2 points) Now, let's consider using this method to look at other objects, like Sather Gate (arch-shaped) and the Greek Theater (bowl-shaped), which are represented in Fig. 3.

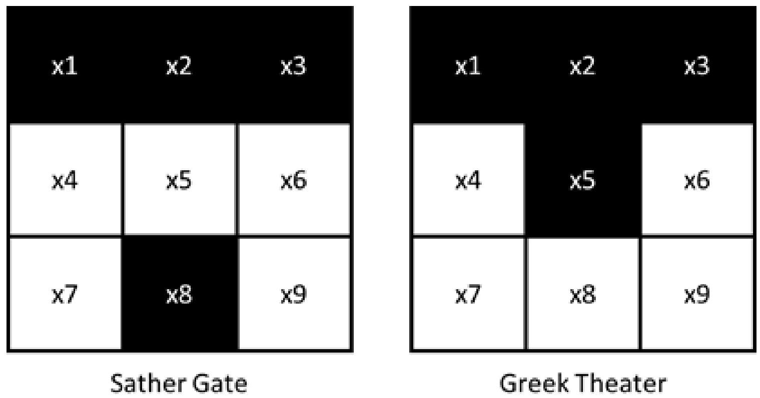



Figure 3: Sather Gate and the Greek Theater represented by 9 pixels

Write down $\vec{x}$ vectors representing these two objects, $\vec{x}_{sather}$ and $\vec{x}_{greek}$, and compute the output vectors $\vec{y}_{sather}$ and $\vec{y}_{greek}$ using the same mask patterns from Fig. 2.

**Solutions:** Similar to previous answers, the $x$-vectors can be written as: $\vec{x}_{sather} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ ; $\vec{x}_{greek} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Similar to previous part, the new output vectors $\vec{y}$ can be quickly computed: $\vec{y}_{sather} = \mathbf{K}\vec{x}_{sather} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ ; $\vec{y}_{greek} = \mathbf{K}\vec{x}_{greek} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

(f) (4 points) Can you use the outputs you computed in the previous part, $\vec{y}_{sather}$ and $\vec{y}_{greek}$, to distinguish between Sather Gate and the Greek Theater? Why or why not?

**Solutions:** No, because the $\vec{y}$ vectors are the same. The $\vec{x}$ vectors for Sather Gate and the Greek Theater are different by two elements, which selects two linearly dependent columns of the $\mathbf{K}$ matrix.

PRINT your name and student ID: ________________________________

(g) (2 points) The difference between the Sather and Greek objects are the pixels $x_5$ and $x_8$. When we generate the vector, $\vec{y}$, the 5th and 8th elements of $\vec{x}$ select the 5th or 8th column of $\mathbf{K}$. In terms of linear independence, what is the relationship between the 5th and 8th column of $\mathbf{K}$?

**Solutions:** They are the same vector, which means they are not linearly independent. (Or, they are linearly dependent on each other.)

(h) (2 points) You want to be able to distinguish all four objects with just two mask patterns, so you try one more thing: turning your illumination source 90° clockwise. This makes the new mask patterns shown in Fig. 4.

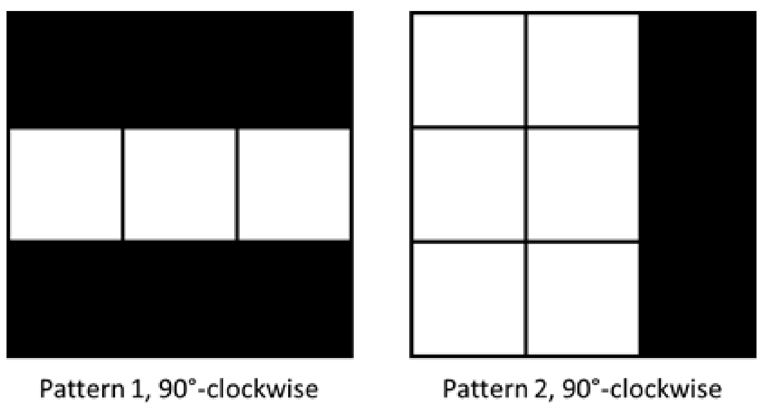



Figure 4: Two scanning patterns, rotated by 90°

Write down the new matrix, $\mathbf{K}_{90}$, representing the new mask patterns. In terms of linear independence, what is the relationship between the 5th and 8th column of $\mathbf{K}_{90}$? Use this knowledge to say whether or not you'll be able to distinguish between Sather Gate and the Greek Theater.

**Solutions:** $\mathbf{K}_{90}$ can be written by inspecting the new source patterns.

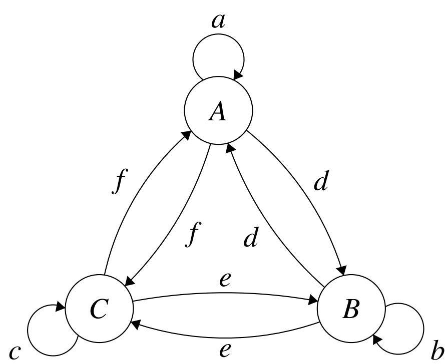
$$\mathbf{K}_{90} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The 5th and 8th columns of $\mathbf{K}_{90}$ are linearly independent, so the output vectors will be distinct. Therefore, we can use these two mask patterns to distinguish Sather Gate and the Greek Theater.

As written, this new $\mathbf{K}$ matrix should be able to distinguish between the Campanile and Cory as well (all four outputs will be distinct).

**8. Reservoirs That Give and Take (15 points)**

Consider a network of three water reservoirs $A$, $B$, and $C$. At the end of each day water transfers among the reservoirs according to the directed graph shown below.




The parameters $a$, $b$, and $c$—which label the self-loops—denote the *fractions* of the water in reservoirs $A$, $B$, and $C$, respectively, that stay in the same reservoir at the end of each day $n$. The parameters $d$, $e$, and $f$ denote the fractions of the reservoir contents that transfer to adjacent reservoirs at the end of each day, according to the directed graph above.

Assume that the reservoir system is conservative—no water enters or leaves the system, which means that the total water in the network is constant. Accordingly, *for each node*, the weights on its self-loop and its two outgoing edges sum to 1; for example, for Node $A$, we have

$$a+d+f=1,$$

and similar equations hold for the other nodes. Moreover, assume that all the edge weights are positive numbers—that is,

$$0<a,b,c,d,e,f<1.$$

The state evolution equation governing the water flow dynamics in the reservoir system is given by $s[n+1]=\mathbf{A}\,s[n]$, where the $3\times 3$ matrix $\mathbf{A}$ is the state transition matrix, and $s[n]=\begin{bmatrix}s_A[n] & s_B[n] & s_C[n]\end{bmatrix}^T\in\mathbb{R}^3$ is the nonnegative state vector that shows the water distribution among the three reservoirs at the end of Day $n$, as fractions of the total water in the network.

In particular, $s[n]\succeq 0$ for all $n=0,1,2,\ldots$, where the symbol $\succeq$ denotes componentwise inequality. Since the state vector represents the fractional distribution of water in the network, we have

$$\vec{1}^T s[n]=\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}s_A[n]\\ s_B[n]\\ s_C[n]\end{bmatrix}=s_A[n]+s_B[n]+s_C[n]=1\quad\forall n=0,1,2,\ldots.$$

(a) (5 points) Determine the state transition matrix $\mathbf{A}$.

**Solutions:**

$$\mathbf{A} = \begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}$$

(b) (5 points) Determine $s^*$, equilibrium state—that is, a state for which the following is true :

$$s[n+1] = s[n] = s^*$$

**Solutions:**

Since the system is conservative, $\vec{1}^T\mathbf{A} = \vec{1}^T$. From above, you can see that $\mathbf{A}^T = \mathbf{A}$. Combining these two facts, $\mathbf{A}\vec{1} = \vec{1}$. This means $\vec{1}$ is the eigenvector that corresponds to $\lambda = 1$, and therefore $\alpha\vec{1}$ is the state that the system will converge to. To ensure $\vec{1}^T s^* = 1$ we let $\alpha = \frac{1}{3}$. So

$$s = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

Another solution is to start from the given conditions. We are told for node 1 that $a+d+f = 1$ and likewise for nodes 2 and 3. Again, we arrive at $\mathbf{A}\vec{1} = \vec{1}$.

Common mistakes in this problem were:

- Found the unit vector to be the solutions, but didn't normalize the final answer such that the sum of the entries was 1.
- Attempts to fin the eigenvector corresponding to the eigenvalue 1 using a method from lecture usually led to algebraically complicated answers.
- Left entries for the equilibrium state in terms of the variable.
- Equilibrium state is a vector, not a matrix.

(c) (5 points) Suppose the state transition matrix for the network is given by

$$\mathbf{A} = \begin{bmatrix} 1/4 & 2/4 & 2/4 \\ 2/4 & 1/4 & 2/4 \\ 2/4 & 2/4 & 1/4 \end{bmatrix}.$$

Is it possible to determine the state $s[n]$ from the subsequent state $s[n+1]$? Provide a reasonably concise, yet clear and convincing explanation to justify your answer. You're NOT asked to compute $s[n]$ from $s[n+1]$ explicitly, but rather to assert, with justification, whether it is possible to do so.

**Solutions:**

We row-reduce $\mathbf{A}$ and see what the pivots look like. If they're all nonzero, then $\mathbf{A}$ is invertible and reverse-time inference (obtaining $s[n]$ from $s[n+1]$) is possible. If even a single pivot is zero, then $\mathbf{A}$ is singular, in which case we cannot determine $s[n]$ from $s[n+1]$.

$$\begin{bmatrix} 1/4 & 2/4 & 2/4 \\ 2/4 & 1/4 & 2/4 \\ 2/4 & 2/4 & 1/4 \end{bmatrix} \sim \begin{bmatrix} 1/4 & 2/4 & 2/4 \\ 0 & -3/4 & -2/4 \\ 0 & -2/4 & -3/4 \end{bmatrix} \sim \begin{bmatrix} 1/4 & 2/4 & 2/4 \\ 0 & -3/4 & -2/4 \\ 0 & 0 & -5/4 \end{bmatrix}$$

All three pivots are nonzero. Therefore $\mathbf{A}$ is invertible and $s[n] = \mathbf{A}^{-1}s[n+1]$. The problem does not ask us to compute $\mathbf{A}^{-1}$ but merely whether it's possible to determine $s[n]$ from $s[n+1]$. The answer is yes!

**9. Eigenvalues of Transition Matrices (15 points)**

(a) (5 points) Show that a square matrix $\mathbf{A}$ and its transpose $\mathbf{A}^T$ have the same eigenvalues.

**Solutions:** We begin with the property that the determinant of a matrix and that of its transpose are equal.

$\det(\lambda\mathbf{I}-\mathbf{A}) = \det\big((\lambda\mathbf{I}-\mathbf{A})^T\big) = \det\big(\lambda\mathbf{I}-\mathbf{A}^T\big)$

Since the two determinants are equal, the characteristic polynomials of the two matrices are also equal. Therefore, they have the same eigenvalues.

(b) (5 points) Let a square matrix $\mathbf{A}$ have rows which sum to one. Show that the vector $\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix}$ is an eigenvector of $\mathbf{A}$ and find its corresponding eigenvalue.

**Solutions:**

Consider the row vector interpretation of matrix vector multiplication. Let $\vec{1}$ denote the vector of all ones. $\mathbf{A}\vec{1} = \vec{1}$, since the rows of $\mathbf{A}$ sum to one. The corresponding eigenvalue is 1.

(c) (5 points) Show that a state transition matrix representing a *conservative* system will always have the eigenvalue $\lambda = 1$. Recall that all columns of a conservative state transition matrix sum to one.

**Solutions:**

Let the matrix $\mathbf{A}$ be a conservative state transtion matrix, i.e. the columns of $\mathbf{A}$ sum to one. Therefore, the rows of $\mathbf{A}^T$ sum to one. From part b), $\mathbf{A}^T$ has the eigenvalue 1, and therefore so does $\mathbf{A}$.

Alternate solution 1: Showing that $\vec{1}^T$ is a left eigenvector and stating that left and right eigenvalues are equal.
$\vec{1}^T \mathbf{A} = \lambda \vec{1}^T$

Alternate solution 2: Showing that $(\mathbf{A} - \mathbf{I})$, when row reduced, will end up with a zero row. This implies that $(\mathbf{A} - \mathbf{I})$ has a non-trivial nullspace, which means that 1 is an eigenvalue.

Common mistakes:
1. Proving this using an example for a 2x2 or 3x3 matrix.
2. Assuming that for a conservative system's matrix, rows also summed to one.
3. Simply stating that a conservative system must have eigenvalue 1 because a steady state exists. This is what the question asked to show. Having an eigenvalue 1 is the reason that a conservative system is able to reach a steady state. The latter cannot be assumed (or simply stated) to prove the former; that is circular logic.
4. Attempt to show that $\vec{1}$ is a right eigenvector.