

## Midterm 1

- **The exam will last 120 minutes.**
- The exam has 4 questions, each worth 25 points, with most of the points being awarded roughly equally between the algorithm and proof. You may be eligible to receive partial credit for your proof even if your algorithm is only partially correct or inefficient.
- Answer all questions. Read them carefully first. Not all parts of a problem are weighted equally.
- Begin each problem on a new page.
- Be precise and concise.
- The problems may **not** necessarily follow the order of increasing difficulty.
- If you use any algorithm from the textbook as a blackbox, you can rely on the correctness of the quoted algorithm. If you modify an algorithm from the textbook, you must explain the modifications precisely and clearly, and if asked for a proof of correctness, give one from scratch or give a modified version of the textbook proof of correctness.
- Good luck!

## 1 Peachy algorithms

Rosa is a peach farmer. She has to tend to  $n$  fields, and all fields are ready for harvest! She must decide in which order to harvest the fields, as she can only harvest one field of peach trees a day. Due to differing soil and geographical conditions, each field has a different water requirement (that stays the same for all days) if it's not harvested, with field  $i$  needing  $\ell_i > 0$  liters of water per day.

- (a) Describe an algorithm that allows Rosa to harvest the fields in such an order that she minimizes the amount of water used by all fields together.
- (b) Prove that your algorithm is correct.
- (c) Analyze the runtime of your algorithm.

## 2 Gameshow

You are participating in a gameshow in which you are presented with a set of  $m$  boxes in a line, each containing an arbitrary unique number. Your goal is to find a box whose number is larger than that of the box to its left and the box to its right (unless it's box 1 or  $m$ , in which case it needs to be larger than just the number of the box it is next to), while opening as few boxes as possible.

- (a) Describe an algorithm to find a box whose number is bigger than that of both of its neighbors. Your solution should run faster than  $O(m)$ .
- (b) Prove that your algorithm is correct.
- (c) Analyze how many boxes your algorithm opens in the worst case. Your final answer should be of the form  $O(f(m))$ , where  $f$  is some function that you specify.

### 3 Greatest Roads Revisited

Quentin wants to travel from San Francisco to New York by bus. Assume that there are  $n - 2$  cities he could change buses at on his way (so  $n$  cities total), with a total of  $m$  direct bus rides between pairs of cities (assume each ride is one-way). The  $i$ th bus ride has price  $p_i$  and all prices are positive integers. Quentin wants to travel as cheaply as possible, while booking a trip such that the sum of all bus ride prices is a multiple of 5 dollars, as he hates small change.

Find an algorithm that finds the cheapest way to get from San Francisco to New York while paying a multiple of 5 dollars for the trip.

- (a) Describe your algorithm.

*Hint: Think about how you can use a similar approach to what you used in a homework problem to solve this question. Furthermore, since all prices are integers, note that there are really only 5 cases you need to consider. It might also help to first consider a case where Quentin wants to end up with an even price.*

- (b) Prove that your algorithm is correct.

*(You can use any algorithm from lecture as a sub-routine, and you do not have to re-argue its correctness.)*

- (c) Analyze the runtime of your algorithm.

### 4 Striped matrices

A *striped matrix* is an  $n$ -by- $n$  matrix  $A$  such that  $a_{ij} = a_{i-1,j-1}$  for  $i = 2, 3, \dots, n$  and  $j = 2, 3, \dots, n$ . For example this is a striped matrix:

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 6 & 4 & 2 & 1 \end{bmatrix}$$

- (a) Show how to represent a striped matrix using  $O(n)$  space instead of the  $O(n^2)$  representation shown above.
- (b) Describe an  $O(n \log n)$ -time algorithm for multiplying an  $n$ -by- $n$  striped matrix  $A$  (represented in your  $O(n)$ -space format from Part (a)) by a vector  $\vec{v}$  of length  $n$ .
- (c) Assume you have a working solution to Part (b). Using that solution, describe an  $O(n^2 \log n)$  algorithm for multiplying an  $n$ -by- $n$  striped matrix  $A$  with an  $n$ -by- $n$  matrix,  $M$ .

*Extra credit (1 pt): Describe a faster algorithm if  $M$  is also a striped matrix. Do not try this unless you have extra time.*