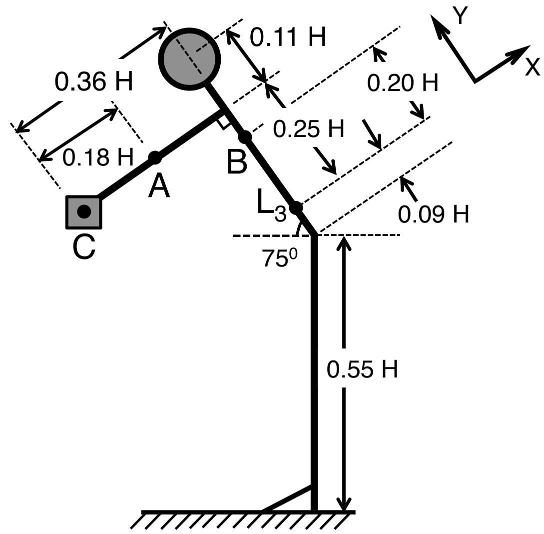
ME C176/BioE C119 Final Exam
Fall 2017 Name:________________________

Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need additional space, use the back sides. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

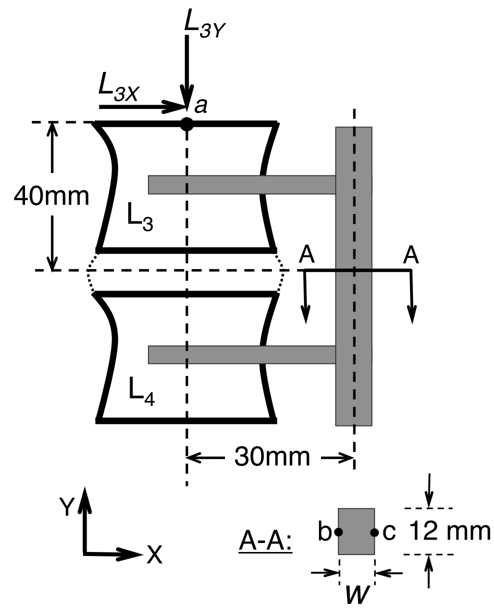
1. [30 points total] Statics and Design of Spine Implants

A person weighing $M = 70\,kg$ and height H bends forward to pick up a $10\,kg$ external mass (assume a point mass, at point C), with arms out straight, as shown. Assume the body above L3 can be modeled as a two-segment system. The first segment comprises the head, neck, and back, and has a mass of $0.44M$ acting at point B; the second segment comprises the arms, and has a mass of $0.1M$ acting at point A. Assume that the abdominal and erector spinae muscles act at a distance of $0.025H$ anterior and posterior to L3, respectively, and their lines of action are parallel to the spinal column.



(i) **[5 points]** Assuming *only one muscle acts*, draw a free body diagram of the body above L3, showing that muscle force and any other forces. Explain why you choose that muscle.

(ii) **[3 points]** Calculate the magnitude of the muscle force.



This person has surgery to support a damaged disc between L3 and L4, using a pedicle screw fixation type of implant, with a vertical rod (having a rectangular cross-section of width W and depth $12\ mm$) rigidly fixed to the pedicle screws; *two identical rods* are used but the planar picture here just shows a lateral view of one of the rods. Assume that the damaged disc cannot transfer any load between the adjacent vertebrae and therefore all load must go through the rods. Assume that the axial (along the spine axis) and transverse components of the L3 force (L_{3Y} and L_{3X} respectively) act at point a in the directions shown and ignore any role of the posterior boney elements in this problem.

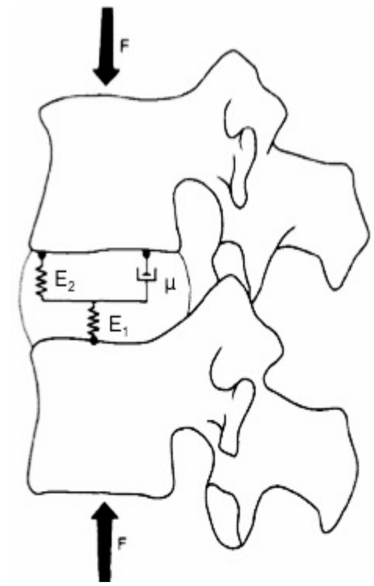
(iii) **[4 points]** Write out an expression for the resultant moment exerted by the top half of each rod on the bottom half of each rod, at section A-A, in terms of L_{3Y} and L_{3X}. *Denote the orientation of this moment.*

(iv) **[8 points]** Write out an expression for the normal stresses in the Y-direction for points b and c at section A–A in terms of W, L_{3Y} and L_{3X}.

(v) **[5 points]** Assuming a maximum allowable tensile or compressive stress in the rod is $350\ MPa$, what is the minimum value of W to avoid failure for these loads? For this calculation, assume that $L_{3X} = 400\ N$, and $L_{3Y} = 2200\ N$.

(vi) **[5 points]** If a bone graft was placed inside the disc in the hopes of generating a fused bone construct between the two vertebrae, why would you want to design the cross-section of the rod to be as small as possible for such applications? Are there any trade-offs or constraints to such an approach? Discuss briefly.

2. [20 Points total] Viscoelasticity



A human cadaver experiment was conducted to determine the compressive mechanical behavior of an L4-L5 spine motion segment. An instantaneous static compressive load of $F = 254\,N$ was applied and maintained constant for a period of $30\,mins$. An instantaneous deformation of $0.96\,mm$ was observed and the deformation reached a steady-state value of $1.26\,mm$. Assume for this system that the intervertebral disc can be modeled as a standard linear solid (as shown in the Figure), the vertebral bodies act like rigid bodies, and the posterior elements do not carry any load. The average cross-sectional area of the disc is $1995\,mm^2$ and average height of the disc is $9.8\,mm$.

(i) [3 Points] What type of test is this typically called? For this test, sketch the general shape of the stress *vs.* time and strain *vs.* time curves assuming the disc behaves like a standard linear solid model.

(ii) [6 Points] Calculate the values of elastic parameters E_1 and E_2.

(iii) [6 Points] The deformation at $10\ min$ is $1.15\ mm$. Calculate the value of the viscosity parameter μ.

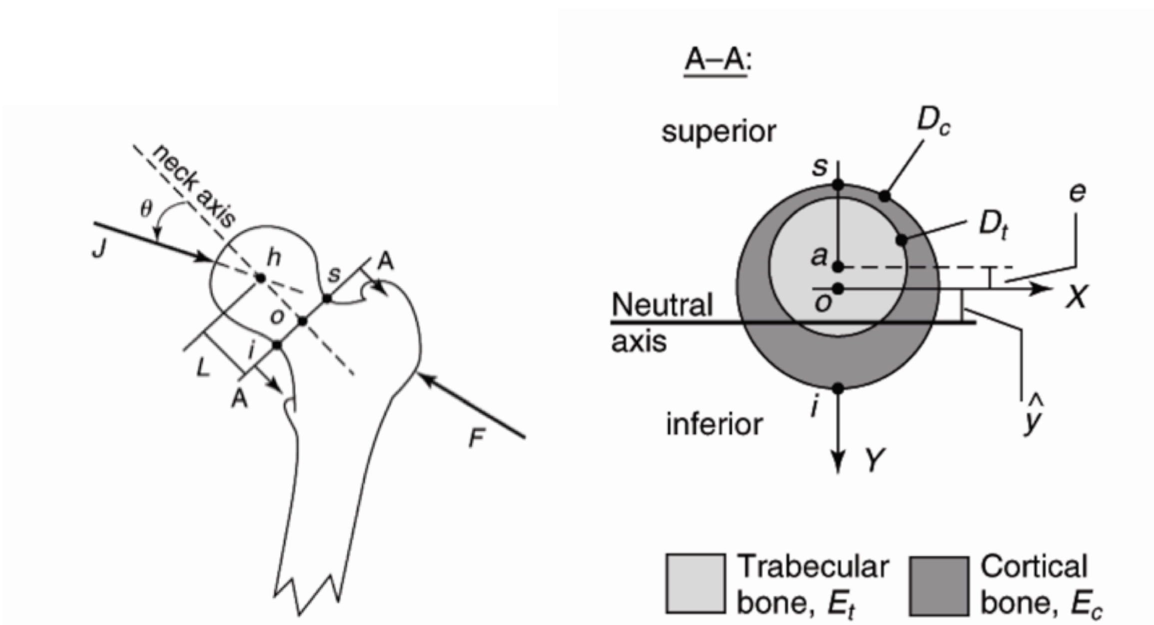
(iv) [5 Points] After this test, the nucleus pulposus portion of the disc was removed and subjected to a sinusoidal cyclic compression test at $1\ Hz$. The measured loss tangent was found to be equal to 0.34. The nucleus pulposus from a second disc specimen from another cadaver reported a loss tangent of 0.16 (for the same loading conditions). Which disc is more degenerated? Explain.

3. [25 Points total] Composite Beam Theory and Hip Fracture Risk

Hip fracture is the most devastating type of osteoporotic fracture. Consider the situation, below, in which the impact force at the side of the hip during a fall is F, and that it develops a joint contact force J, at some angle θ as well as some equilibrating loads acting on the distal femur (not shown).

Section A-A is a schematic of the narrowest cross-section of the femoral neck. The cross-section contains circular areas representing the cortical bone (modulus E_c, center o, outer diameter D_c) and the trabecular bone (modulus E_t, center a, outer diameter D_t) with the trabecular region displaced superiorly by a distance e. The X-Y coordinate system is centered at origin o, and the neutral axis of the composite beam is at a distance $\hat{y}$ below the X-axis.

The angle of the joint contact force, θ , plays an important role in hip fracture etiology. The amount of eccentricity, e, between the cortical and trabecular bone also plays an important role. Let's investigate how the stress at the inferior-most point, i, on the femoral neck depends on θ and e.



The stress acting on the inferior-most point i, can be expressed by the following composite beam theory-based equation:

$$\sigma = -\frac{PE_c}{\sum_i E_i A_i} + \frac{E_c M t}{\sum_i E_i \hat{I}_i}$$

in which P is the axial force acting on section A-A, M is the total bending moment acting on section A-A (this is the moment acting *about the neutral axis*), t is the distance from point i to the neutral axis of section A-A, and the summation terms $\sum_i E_i A_i$ and $\sum_i E_i \hat{I}_i$ represent the axial and flexural stiffnesses, respectively, of the composite section, the latter with respect to the neutral axis.

Using the above diagram and stress equation, write out expressions for the following terms as a function only of the variables indicated:

(i) **[5 points]** $P = f(J, \theta) =$

(ii) **[5 points]** $M = f(J, \theta, L, \hat{y}) =$

(iii) **[7 points]** $t = f(D_c, D_t, e, E_c, E_t) =$

(iv) **[8 points]** $\sum_i E_i \hat{I}_i = f(D_c, D_t, e, E_c, E_t, \hat{y}) =$

 Name:________________________

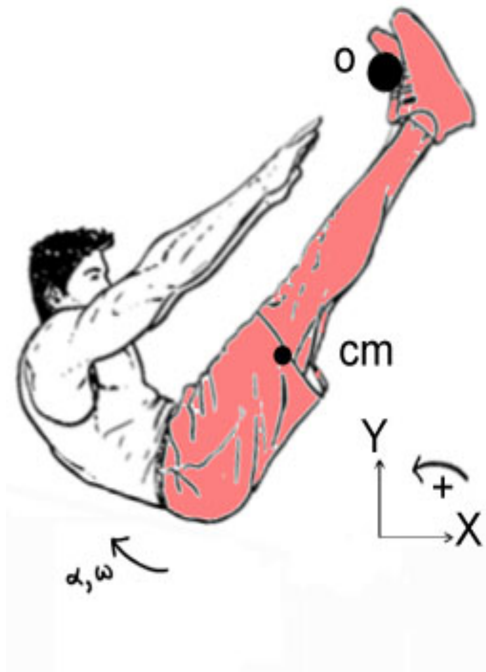
4. [25 Points] Miscellaneous

(i) [10 points]
For the following table, give an approximate value of the elastic modulus for each of the following materials and human tissues.

Material	Elastic Modulus (MPa)
Ligament	
Tendon	
Articular cartilage (static modulus)	
Trabecular bone (spine, osteoporotic)	
Cortical bone (femur, longitudinal)	
UHMWPE	
PMMA (bone cement)	
Co-Cr alloy (forged)	
Co-Cr alloy (cast)	
Ti-4Al-6V alloy (annealed)	

(ii) [5 points] Briefly explain why the UHMWPE plastic fails mainly by abrasion at the hip but by cracking and fracture at the knee.

(iii) [10 points] Planar motion. At this instant, a gymnast is swinging in a circular arc around a fixed rod (point O), at angular velocity ω and angular acceleration α as shown, and his ankle and knee joints are both rigid. Assume this is a 2D problem. The mass of the lower body (shaded portion) is M_{LB}, its center of mass is at point **cm**, and its moment of inertia about point O is I_{LB}.

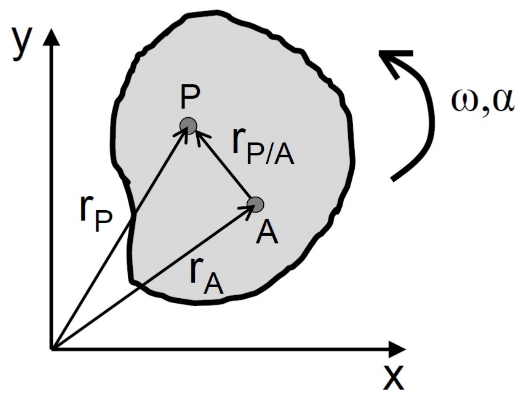


a. Draw a fully labeled free body diagram of the lower body (shaded portion) of this gymnast. Include all loads and accelerations.

b. Write out the moment equation of motion corresponding to your free body diagram. Make sure any dimensions or other quantities used in your equation are clearly shown and labeled in your free body diagram. Use the sign convention shown to denote positive values, assuming moments are postive *counter-clockwise*.

Orthopedic Biomechanics
Fall 2017 Formula Sheet

1 Dynamics (Planar Rigid Body Motion)



$\underline{a}_P = \underline{a}_A + \underbrace{(\underline{\alpha} \times \underline{r}_{P/A})}_{a_{tangential}} \underbrace{-(\omega^2 \underline{r}_{P/A})}_{a_{radial}}$ (Relative acceleration between two points on a rigid body)

Moment Balance about center of mass, cm

$\sum \underline{M}_{cm} = I_{cm}\underline{\alpha}$ (I_{cm} = moment of inertia about the center of mass of the system)

Moment Balance about a fixed point O

$\sum \underline{M}_O = I_O\underline{\alpha}$ (I_O = moment of inertia about the fixed point O)

Parallel axis theorem: $I_O = (I_{cm} + mr^2_{cm/O})$

Moment Balance about an accelerating point P

$\sum \underline{M}_P = I_{cm}\underline{\alpha} + (\underline{r}_{cm/P} \times m\underline{a}_{cm})$

2 Impulse Momentum

$$\int_{t_i}^{t_f} (\sum \underline{F})dt = \underline{p}_f - \underline{p}_i$$

where $\sum \underline{F}$ is the net sum of the external forces, $\underline{p} = m\underline{v}$ is the momentum

3 Potential and Kinetic Energy

Linear Kinetic Energy

K.E.$= \frac{1}{2}mv^2$ (m = mass and v = velocity)

Rotational Kinetic Energy around Fixed Point P

K.E.$_P = \frac{1}{2}I_p\omega^2$ (I_P = Moment of inertia around P, ω = angular velocity)

Gravitational Potential Energy

P.E. $= mgh_{cm}$ (m = mass, g = gravitational constant, h_{cm} = height of center of mass)

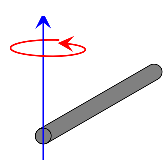
4 List of mass moment of inertia

Point Mass

$I = mr^2$ (m = mass, r = distance to point)

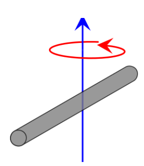
Slender rod around fixed end point P

$I_p = \frac{ml^2}{3}$ (m = mass, l = length)



Slender rod around its center of mass

$I_{cm} = \frac{ml^2}{12}$ (m = mass, l = length)



5 Vector operators

$\underline{A} \cdot \underline{B} = |\underline{A}||\underline{B}|\, cos(\theta) = A_1B_1 + A_2B_2 + A_3B_3$

$$\begin{aligned} \underline{A} \times \underline{B} &= |\underline{A}||\underline{B}| \sin(\theta)\, \hat{\boldsymbol{n}} \\ &= (A_2B_3 - A_3B_2)\hat{\boldsymbol{\imath}} + (A_3B_1 - A_1B_3)\hat{\boldsymbol{\jmath}} + (A_1B_2 - A_2B_1)\hat{\boldsymbol{k}} \end{aligned}$$

6 Viscoelasticity

Elastic Spring : $\sigma = \mathrm{E}\epsilon$
Viscoelastic Dashpot : $\sigma = \mu\dot{\epsilon}$

Creep (c(t)) and Stress Relaxation (s(t)) Functions

$$\mathrm{c(t)_M} = \frac{1}{\mathrm{E}} + \frac{1}{\mu}\mathrm{t} \qquad \mathrm{s(t)_M} = \mathrm{E}\mathrm{e}^{-\mathrm{t}/\tau_\mathrm{R}} \qquad \tau_\mathrm{R} = \frac{\mu}{\mathrm{E}}$$

$$\mathrm{c(t)_{KV}} = \frac{1}{\mathrm{E}}(1 - \mathrm{e}^{-\mathrm{t}/\tau_\mathrm{R}}) \qquad \mathrm{s(t)_{KV}} = \mu\delta(\mathrm{t}) + \mathrm{E}$$

$$\mathrm{c(t)_{SLS}} = \frac{1}{\mathrm{E_1}} + \frac{1}{\mathrm{E_2}}(1 - \mathrm{e}^{-\mathrm{t}/\tau_1}) \qquad \mathrm{s(t)_{SLS}} = \frac{\mathrm{E_1E_2}}{\mathrm{E_1+E_2}}\left\{1 + \frac{E_1}{E_2}e^{-t/\tau_2}\right\} \qquad \tau_1 = \frac{\mu}{\mathrm{E_2}} \qquad \tau_2 = \frac{\mu}{\mathrm{E_1+E_2}}$$

Principle of linear superposition/ Hereditary Integral

$$\epsilon(\mathrm{t}) = \sum \sigma_\mathrm{i}\mathrm{c}(\mathrm{t} - \mathrm{t_i})$$

$$\epsilon(\mathrm{t}) = \int_{-\infty}^{\mathrm{t}} \frac{\mathrm{d}\sigma(\tau)}{\mathrm{d}\tau}\mathrm{c}(\mathrm{t} - \tau)d\tau$$

Sinusoidal Loading

$$\sigma(\mathrm{t}) = \sigma_0\mathrm{sin}\omega\mathrm{t},\ \ \epsilon(\mathrm{t}) = \epsilon_0\mathrm{sin}(\omega\mathrm{t} - \delta),\ \ \mathrm{E_S} = \frac{\sigma_0\mathrm{cos}\delta}{\epsilon_0}\ \text{(Storage Modulus)},\ \ \mathrm{E_L} = \frac{\sigma_0\mathrm{sin}\delta}{\epsilon_0}\ \text{(Loss Modulus)}$$

$$\mathrm{tan}\delta\ \text{(Loss Tangent)}$$

7 Hertz Contact Theory

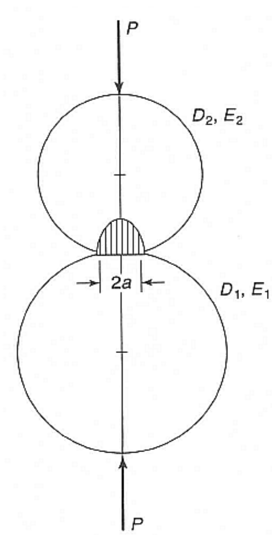
$$\mathrm{p_{max}} = \frac{1.5\mathrm{P}}{\pi\mathrm{a}^2}$$

$$\mathrm{a} = 0.721(\mathrm{PC_GC_M})^{1/3}\ \ \text{where}\ \ \mathrm{C_M} = \frac{1-\nu_1^2}{\mathrm{E_1}} + \frac{1-\nu_2^2}{\mathrm{E_2}}$$

$$\mathrm{C_G} = \frac{\mathrm{D_1D_2}}{\mathrm{D_1+D_2}}\ \text{(convex contact)}$$

$$\mathrm{C_G} = \mathrm{D_2}\ \text{(flat contact)}$$

$$\mathrm{C_G} = \frac{\mathrm{D_1D_2}}{\mathrm{D_1-D_2}}\ \text{(concave contact)}$$



$$\sigma_{\mathrm{C_{max}}} = \frac{1.5\mathrm{P}}{\pi\mathrm{a}^2}$$

$$\tau_\mathrm{max} = \frac{\sigma_\mathrm{max}}{3}\ \ \text{at a depth of 0.51a}$$

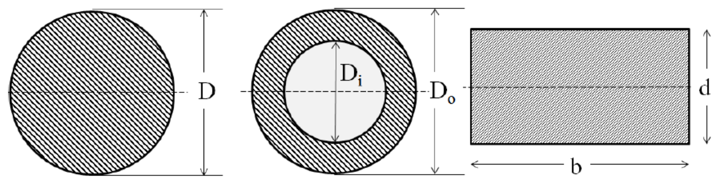
$$\sigma_{\mathrm{T_{max}}} = \frac{(1-2\nu)\sigma_\mathrm{max}}{3}\ \text{at radius a}$$

$$\sigma_\mathrm{vonMises} = \sqrt{\frac{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}{2}}$$

$\sigma_1, \sigma_2, \sigma_3$ are principal stresses

8 Symmetric Beam Theory

$$I = \frac{\pi D^4}{64} \qquad I = \frac{\pi(D_o^4 - D_i^4)}{64} \qquad I = \frac{bd^3}{12}$$



(i) Simple Beam - Axial Loading

$$\sigma = \frac{P}{A}$$

(ii) Simple Beam - Bending

$$\sigma = -\frac{Mt}{I}$$

(iii) Composite Beam - Axial Loading

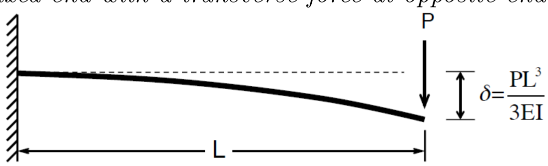
$$\sigma_j = \frac{E_j P}{\sum_{i=1}^{n} E_i A_i}$$

(iv) Composite Beam - Bending

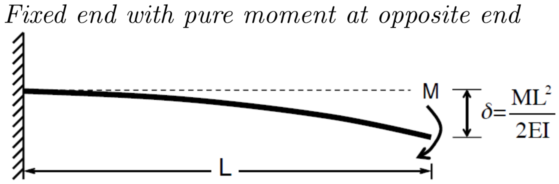
$$\sigma_j = \frac{-E_j M t}{\sum_{i=1}^{n} E_i \hat{I}_i} \quad \text{where} \quad \hat{I}_i = \bar{I}_i + A_i(\bar{y}_i - \hat{y})^2 \ , \ \hat{y} = \frac{\sum_{i=1}^{n} E_i A_i \bar{y}_i}{\sum_{i=1}^{n} E_i A_i}$$

8 Deflection of Beams

Fixed end with a transverse force at opposite end



Fixed end with pure moment at opposite end



9 Asymmetric Beam Theory

$$\sigma_i = E_i \frac{(M_y I_{zz}^* \ + \ M_z I_{yz}^*)s \ - \ (M_y I_{yz}^* \ + \ M_z I_{yy}^*)t}{I_{yy}^* I_{zz}^* \ - \ (I_{yz}^*)^2} \quad \text{where} \ \ t = y - \hat{y}, \ \ s = z - \hat{z}$$

$$I_{yy}^* \ = \ \sum_{i=1}^{n} E_i \left[\bar{I}_{yy_i} \ + \ (\bar{z}_i - \hat{z})^2 \ A_i \right], \qquad \hat{y} = \frac{\sum_{i=1}^{n} E_i A_i \bar{y}_i}{\sum_{i=1}^{n} E_i A_i}$$

$$I_{zz}^* \ = \ \sum_{i=1}^{n} E_i \left[\bar{I}_{zz_i} \ + \ (\bar{y}_i - \hat{y})^2 \ A_i \right], \qquad \hat{z} = \frac{\sum_{i=1}^{n} E_i A_i \bar{z}_i}{\sum_{i=1}^{n} E_i A_i}$$

$$I_{yz}^* \ = \ \sum_{i=1}^{n} E_i \left[\bar{I}_{yz_i} \ + \ (\bar{y}_i - \hat{y})(\bar{z}_i - \hat{z}) A_i \right]$$

$$\tan\alpha = \left[\frac{M_y I_{zz}^* \ + \ M_z I_{yz}^*}{M_y I_{yz}^* \ + \ M_z I_{yy}^*} \right]$$