Physics 7B Lecture 2 Midterm 1 Problem 1

Taige Wang

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1. When we talked about thermal expansion, we chose to reference the change in the length of a rod by its length L_0 at its initial temperature T_0 :

$$\Delta L = L_0 \alpha \Delta T$$

from which we found that the length of the rod at temperature T is

$$L = L_0(1 + \alpha \Delta T)$$

Here, $\Delta L = L - L_0, \Delta T = T - T_0$, and α is the linear coefficient of expansion.

(a) Suppose we chose to reference the change in the length of the rod by its length L at temperature T instead of L_0 so that

$$\Delta L = L\tilde{\alpha}\Delta T$$

and $\tilde{\alpha}$ is the linear coefficient of expansion with this choice. What is L in terms of $L_0, \Delta T$, and $\tilde{\alpha}$ with this choice?

Solution: (6 pts.) Since $\Delta L = L - L_0$, we have

$$L - L_0 = L\tilde{\alpha}\Delta T \qquad \text{(2 pts.)} \tag{1}$$

We can reorganized the equation as

$$L(1 - \tilde{\alpha}\Delta T) = L_0 \implies L = \frac{L_0}{1 - \tilde{\alpha}\Delta T} \qquad \text{(4 pts.)} \tag{2}$$

(b) What is $\tilde{\alpha}$ in terms of α and ΔT?

Solution: (7 pts.) Now we compare this new expansion formula with the old one,

$$L = L_0(1 + \alpha\Delta T) = \frac{L_0}{1 - \tilde{\alpha}\Delta T} \qquad \text{(3 pts.)} \tag{3}$$

After some algebra we find

$$1 - \tilde{\alpha}\Delta T = \frac{1}{1 + \alpha\Delta T} \implies \tilde{\alpha} = \frac{1}{\Delta T}\left(1 - \frac{1}{1 + \alpha\Delta T}\right) = \frac{\alpha}{1 + \alpha\Delta T} \qquad \text{(4 pts.)} \tag{4}$$

(c) The expression obtained in part a for L should be different from Eq. (1) above. Given the typical size of α, are these differences significant? Justify your answer in detail. (You might find the approximation $(1+u)^p \approx 1 + pu$ for $u \ll 1$ useful.)

Solution: (7 pts.) To compare these two formulas, we actually need to compare α and $\tilde{\alpha}$. We first Taylor expand $\tilde{\alpha}$ in terms of α,

$$\tilde{\alpha} = \alpha(1 + \alpha\Delta T)^{-1} \approx \alpha(1 - \alpha\Delta T) \qquad \text{(4 pts.)} \tag{5}$$

Since α is small, $\alpha^2\Delta T$ is extremely small for almost all materials. Thus, the differences are not significant (these types of terms get ignored when calculating the volumetric expansion) (3 pts.).

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2. (a) (8 points) When the system reaches equilibrium, we have ice and liquid water coexisting in the container, both at the same temperature T. This can only happen at the phase transition temperature, which for water is 0 °C.

$$T = 0\,^\circ\text{C} \qquad \text{(2 points)} \tag{1}$$

The explanation is worth 6 points.

- 6 points: clear, complete, correct explanation
- 5 points: one minor error/omission
- 4 points: multiple minor errors/omissions or one major error/omission
- 3 points: multiple major errors/omissions
- 2 points: only one clear, correct, relevant statement
- 1 point: nothing that would lead to a correct answer
- 0 points: no explanation

Partial credit was be given for calculating T from m_{ice}, c_{ice}, T_{ice}, m_w, c_w, and T_w. Students cannot receive points for this if they received any of the previous points.

$$0 = Q_{ice} + Q_w \tag{2}$$
$$= m_{ice}c_{ice}\left(T - T_{ice}\right) + m_w c_w \left(T - T_w\right) \tag{3}$$
$$T = \frac{m_{ice}c_{ice}T_{ice} + m_w c_w T_w}{m_{ice}c_{ice} + m_w c_w} \qquad \text{(3 points)} \tag{4}$$

(b) (12 points) Let Q_{ice} be the heat absorbed by the ice and Q_w the heat absorbed by the liquid water (note that this will be negative, because the water lost heat). The total heat absorbed by the system is $Q_{net} = Q_{ice} + Q_w$. Since the system is thermally isolated, there is no exchange of energy with the surroundings, ie $Q_{net} = 0$. Since no ice melted and no liquid water froze, there will be no latent heat (2 points). This means that all of the heat absorbed by ice and the liquid water will each be given by

$$Q = mc\Delta T \qquad \text{(2 points).} \tag{5}$$

Putting everything together, we get

$$0 = Q_{net} \tag{6}$$
$$= Q_{ice} + Q_w \qquad \text{(2 points)} \tag{7}$$
$$= m_{ice}c_{ice}\left(T - T_{ice}\right) + m_w c_w \left(T - T_w\right) \qquad \text{(4 points)} \tag{8}$$
$$\frac{m_w}{m_{ice}} = \frac{c_{ice}\left(T - T_{ice}\right)}{c_w\left(T_w - T\right)} \tag{9}$$
$$= -\frac{c_{ice}T_{ice}}{c_w T_w} \qquad \text{(2 points)} \tag{10}$$

where we have used $T = 0\,^\circ\text{C}$ in the last line. Because our formula depends only on temperature differences, we can do everything in Celsius. Since we have use T in Celsius, our final answer is only valid with T_{ice} and T_w in Celsius. Note that the mass ratio is positive, because T_{ice} is negative in Celsius.

3. Consider a monatomic gas in a container. As we are interested only in motion along the x-direction, take the probability of finding a particle with velocity between v_x and $v_x + dv_x$ to be,

$$\mathrm{d}P = N\sqrt{\frac{m}{2\pi k_B T}}e^{-mv_x^2/2k_BT}\,\mathrm{d}v_x \tag{1}$$

and the average of any quantity $B(v_x)$ per particle to be,

$$\langle B(v_x)\rangle = \frac{1}{N}\int_{-\infty}^{\infty} B(v_x)\,\mathrm{d}P \tag{2}$$

Here, m is the mass of a particle that makes up the gas.

(a) **We can represent the velocity of a particle along the positive x-axis to be:**

$$v_x^+ \equiv \begin{cases} v_x & \text{if } v_x > 0 \\ 0 & \text{if } v_x \leq 0 \end{cases} \tag{3}$$

What is $\langle v_x^+\rangle$? Express it in terms of m, T, and k_B.
Solution: The definition of the average of an arbitrary function implies,

$$\left\langle v_x^+ \right\rangle = \frac{1}{N}\int_{-\infty}^{\infty} v_x^+ N\sqrt{\frac{m}{2\pi k_B T}}e^{-mv_x^2/2k_BT}\,\mathrm{d}v_x \tag{4}$$

However, since v_x^+ is 0 for $v_x \leq 0$, the integral can be done from 0 to ∞ instead. After that, it is a matter of calculating,

$$\left\langle v_x^+ \right\rangle = \frac{1}{N}\int_{0}^{\infty} v_x^+ N\sqrt{\frac{m}{2\pi k_B T}}e^{-mv_x^2/2k_BT}\,\mathrm{d}v_x \tag{5}$$

$$= \sqrt{\frac{m}{2\pi k_B T}}\int_{0}^{\infty} v_x e^{-mv_x^2/2k_BT}\,\mathrm{d}v_x \tag{6}$$

$$= \sqrt{\frac{m}{2\pi k_B T}}\frac{2k_BT}{m}\int_{0}^{\infty} ue^{-u^2}\,\mathrm{d}u \tag{7}$$

$$= \sqrt{\frac{2k_BT}{m\pi}}\int_{0}^{\infty} ue^{-u^2}\,\mathrm{d}u \tag{8}$$

where in the second to last line, we have used a substitution

$$u^2 = \frac{mv_x^2}{2k_BT} \implies v_x\,\mathrm{d}v_x = \frac{2k_BT}{m}u\,\mathrm{d}u \tag{9}$$

This is a Gaussian integral that can be done via Feynman's trick or by looking at the integral table. The integral table. There it says,

$$\int_{0}^{\infty} u^{2n+1}e^{-u^2}\,\mathrm{d}u = \frac{1}{2}n! \tag{10}$$

where we use the $n = 1$ case. Substituting that into our equation, we get,

$$\left\langle v_x^+ \right\rangle = \sqrt{\frac{k_bT}{2\pi m}} \tag{11}$$

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(b) **What is $\langle (v_x^+)^2 \rangle$? Express it in terms of m, T, and k_B.**
Solution: The definition of the average of an arbitrary function implies,

$$\left\langle \left(v_x^+\right)^2 \right\rangle = \frac{1}{N} \int_{-\infty}^{\infty} (v_x^+)^2 N \sqrt{\frac{m}{2\pi k_B T}} e^{-mv_x^2/2k_BT} \, \mathrm{d}v_x \tag{12}$$

However, since v_x^+ is 0 for $v_x \leq 0$, the integral can be done from 0 to ∞ instead. After that, it is a matter of calculating,

$$\left\langle \left(v_x^+\right)^2 \right\rangle = \frac{1}{N} \int_{0}^{\infty} (v_x)^2 \, N \sqrt{\frac{m}{2\pi k_B T}} e^{-mv_x^2/2k_BT} \, \mathrm{d}v_x \tag{13}$$

$$= \sqrt{\frac{m}{2\pi k_B T}} \int_{0}^{\infty} (v_x)^2 \, e^{-mv_x^2/2k_BT} \, \mathrm{d}v_x \tag{14}$$

$$= \sqrt{\frac{m}{2\pi k_B T}} \left(\frac{2k_BT}{m}\right)^{3/2} \int_{0}^{\infty} u^2 e^{-u^2} \, \mathrm{d}u \tag{15}$$

$$= \frac{2k_bT}{m\sqrt{\pi}} \int_{0}^{\infty} u e^{-u^2} \, \mathrm{d}u \tag{16}$$

where in the second to last line, we have used a substitution

$$u^2 = \frac{mv_x^2}{2k_BT} \implies v_x^2 \, \mathrm{d}v_x = \left(\frac{2k_BT}{m}\right)^{3/2} u \, \mathrm{d}u \tag{17}$$

This is a Gaussian integral that can be done via Feynman's trick or by looking at the integral table. The integral table. There it says,

$$\int_{0}^{\infty} u^{2n} e^{-u^2} \, \mathrm{d}u = \frac{\sqrt{\pi}}{2^{n+1}} \frac{(2n-1)!}{n!} \tag{18}$$

where we use the $n = 1$ case. Substituting that in,

$$\left\langle \left(v_x^+\right)^2 \right\rangle = \frac{k_bT}{2m} \tag{19}$$

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20 points

- Set up Eq. 4 gets 5 points
- Simplify to Integral Eq. 8 gets 3 points
- Answer Eq. 11 gets 2 points
- Set up Eq. 13 gets 5 points
- Simplify to Integral Eq. 15 gets 3 points
- Answer Eq. 19 gets 2 points

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6 pts

3pts for correct answer + 3pts for correct explanation

a. The temperature of the iron sphere will never drop below T_{CMB}. As the iron cools, the rate at which it loses energy through radiation decrease. The rate at which the iron gains energy from the CMB is constant, because it is determined by T_{CMB}. When the temperature of the iron reaches T_{CMB}, the rate at which it loses energy through radiation and gains energy from the CMB exactly match, keeping the temperature of the iron constant.

14 pts

b. $$\frac{dQ}{dt} = \varepsilon \sigma A \left(T_{CMB}^4 - T^4 \right)$$

↗ absorption ↑ emission

3pts for writing down Stefan-Boltzmann law

2pts for the correct expression for dQ/dt

$$dQ = mc\, dT$$

3pts for dQ=mcdT

$$\int_{T_0}^{T_f} \frac{dT}{T_{CMB}^4 - T^4} = \int_{t_0}^{t_f} \frac{\varepsilon \sigma A}{mc}\, dt$$

$$T_0 = 2^7 T_{CMB}, \quad T_f = 2 T_{CMB}$$

2pts for the differential equation for t and T

$$\frac{\varepsilon \sigma A T_{CMB}^3}{mc} \Delta t = \int_{2^7}^{2} \frac{dx}{1 - x^4}$$

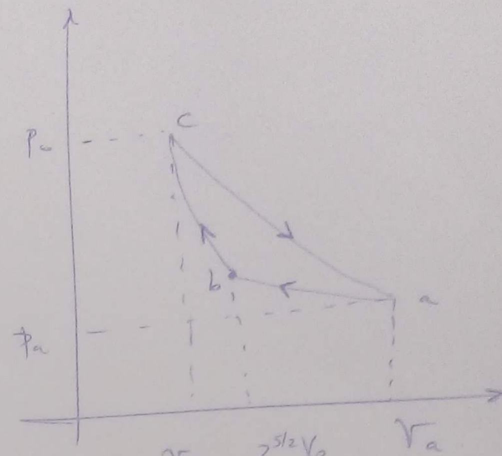
2pts for the integral

$$= \frac{1}{2} \left[\operatorname{atan}(2) - \operatorname{atan}(2^7) + \operatorname{atanh}(2) - \operatorname{atanh}(2^7) \right] \approx .0428$$

$$\boxed{\Delta t = .0428 \frac{m\, c_{Fe}}{e_{Fe}\, \sigma_{SB}\, A} \frac{1}{T_{CMB}^3}}$$

2 pts for final answer

Problem 5 $\gamma = 7/5$



(a) $T_a = T_b$ since $a \to b$ is isothermal.

$T_b \neq T_c$ since $b \to c$ is adiabatic.

$\left(\frac{5}{2} nR \Delta T = dW \right)$ ← compression

$\longrightarrow$ $T_c \neq T_a$.

$\therefore$ $c \to a$ can not be isothermal.

(b) $c \to a$: $P = P_0 \left(\frac{V}{V_0} \right)^{-6/5}$ . . . (1)

$b \to c$ is adiabatic. So: $P_b V_b^{\gamma} = P_c V_c^{\gamma}$ . . . (2)

$a \to b$ is isothermal. So: $P_a V_a = P_b V_b$ . . . (3)

from (2): $P_b \left(2^{5/2} V_0 \right)^{7/2} = P_0 V_0^{7/2} \longrightarrow P_b = 2^{-7/2} P_0$

from (3): $P_a V_a = \left(2^{-7/2} P_0 \right)\left(2^{5/2} V_0 \right) = \frac{1}{2} P_0 V_0$ . . . (4)

Using (4) and (1):

$V_a = 2^5 V_0$

$P_a = 2^{-6} P_0$.

(c)

$Q_{cycle} = Q_{ab} + Q_{bc} + Q_{ca}$.

Q_{ab}: Isothermal, W_{ab}; $Q_{bc} = 0$ ← adiabatic.

- $Q_{ab} = NKT_a \ln \left(\frac{V_b}{V_a} \right) = -\frac{5}{4} P_0 V_0 \ln(2)$.

- $Q_{ca} = \Delta E_{ca} + W_{ca}$

$\Delta E_{ca} = \frac{d}{2} NK (T_a - T_c) = -\frac{5}{4} P_0 V_0$.

$W_{ca} = \int p\,dV = \int_{V_c}^{V_a} P_0 V_0^{6/5} V^{-6/5}\,dV$

$= 5 P_0 V_0$

- the efficiency: $e = \frac{W}{Q_H} \longrightarrow Q_H = Q_{ca} = \frac{5}{4} P_0 V_0$.

$\Delta \overset{0}{E} = Q - W \Rightarrow W = Q_{cycle} = \frac{5}{4} P_0 V_0 \left(1 - \ln(2)\right)$.

$\therefore \quad e = 1 - \ln(2)$

L2 MT1 Problem 5 Rubric

a)

.Correct Analysis (+3 points)

b)

.Correct Relation for Adiabatic Process (b ->c) (+1 point)

.Correct Relation for Isothermal Process (a->b) (+1 point)

.Correct Relation for The given Process (c->a) (+1 point)

.Correct Answer (+4 points)

.Minor Mistake (-1 point)

c)

.Correct Heat for Adiabatic Process (b ->c) (+1 point)

.Correct Heat for Isothermal Process (a->b) (+2 point)

.Correct Heat for The given Process (c->a) (+3 point)

.Correct Answer (+1 point)

.Minor Mistake (-1 point)

d)

. Correct Work (+1 point)

. Correct Heat (+1 point)

. Correct Answer (+1 point)

.Minor Mistake (-1 point)