Midterm Exam

Math H113, Feb. 25, 2021. Instructor: E. Frenkel

You have 1 hour and 10 minutes to solve the following equally-weighted problems.

You will then have up to 20 minutes to upload your exam to Gradescope.

You have to upload your exam to Gradescope by 2 pm.

You are not allowed to use any materials during the exam. You are not allowed to discuss the exam with anyone during the exam.

At the top of the first page of your exam, please write clearly your name and SID number, then write the following pledge and **sign your name under it**:

"The work on this exam is entirely my own, I have worked on it for no more than 1 hour and 10 minutes, and I have not assisted other students in any way during this exam."

If you have a question about the exam problems (or having difficulties with uploading your exam to Gradescope), send an email to E. Frenkel (frenkelmath@gmail.com).

If the instructor needs to communicate something during the exam (for example, if there is a typo or if a clarification is needed), **he will send an email to all students. So please keep an eye on your email messages.**

Please follow these rules:

- **Make sure your submission is legible**. Points will be deducted for illegible submissions.
- Separate your solutions to different problems by a horizontal line.
- Write your solutions in the order in which the problems appear.
- Write the number of each problem clearly.
- Encircle your final answers.
- **Justify your answers.** You will not get full credit for unsubstantiated or unjustified answers.

Problem 1.

Consider the group $\mathbb{Z}_{24}$.

(a) Describe its subgroup generated by the element 15.
(b) Give the list of all elements x of this group with the following property: the cyclic subgroup generated by x is isomorphic to $\mathbb{Z}_4$.
(c) Draw the diagram of all subgroups of $\mathbb{Z}_{24}$.

Problem 2. Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 9 & 8 & 4 & 3 & 1 & 2 & 6 \end{pmatrix}$$

(a) Describe the orbits of σ.
(b) Express σ as a product of disjoint cycles, and then as a product of transpositions.
(c) What is the order of σ? Explain.

Problem 3. Let G be a group.

(a) Given two elements $a, b \in G$, define $\phi_{a,b} : \mathbb{Z} \times \mathbb{Z} \to G$ by the formula
$$\phi_{a,b}(m, n) = a^m b^n, \qquad m, n \in \mathbb{Z}.$$
Give the necessary and sufficient conditions on a and b for $\phi_{a,b}$ to be a group homomorphism, and prove that this is so.
(b) For a positive integer k, define the group $\mathbb{Z}^k$ by induction: $\mathbb{Z}^k = \mathbb{Z} \times \mathbb{Z}^{k-1}$ for $k > 1$, and $\mathbb{Z}^1 = \mathbb{Z}$. Give an explicit description of the set of all homomorphisms $\phi : \mathbb{Z}^k \to G$ in terms of the group G (do not just give the definition) and prove it.

Problem 4. For each group H below, determine whether the symmetric group S_5 has a subgroup isomorphic to H. If yes, then give an example of such a subgroup. If no, explain why not.

(a) $H = \mathbb{Z}_5$
(b) $H = \mathbb{Z}_6$
(c) $H = \mathbb{Z}_7$

Problem 5. Let G be a group.

(a) Suppose that H is a subgroup of G of index 2. Prove that H is a normal subgroup.
(b) Suppose that H is a subgroup of G of index 3. Either prove that H is a normal subgroup or give a counterexample and explain why it is a counterexample.

Problem 6. An *automorphism* of a group G is a permutation $f : G \to G$ which is a group isomorphism.

(a) Prove that the set of all automorphisms of a given group G is a subgroup of the group S_G of all permutations of G. Denote it by $\mathrm{Aut}(G)$.
(b) Describe $\mathrm{Aut}(\mathbb{Z})$.
(c) Describe $\mathrm{Aut}(\mathbb{Z}_{12})$.

Problem 7. Describe the group of automorphisms of the symmetric group S_3.

Note: In parts (b) and (c) of Problem 6 and in Problem 7, "describe" means describing the group *and* identifying it with a group we have previously studied.