Department of EECS - University of California at Berkeley

EECS 126 - Probability and Random Processes - Fall 2008

Midterm 1: 10/09/2008

Name:

SID:

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1. Short Questions (20%); 4% each

1.1. Define "Random Variable"

Answer:

A random variable is a real-valued function of the outcome or a random experiment (80%). Details: To specify the random experiment, one defines a probability space (set of outcomes Ω and probability $P(A)$ of sets A of outcomes). The random variable is then a function $X : \Omega \to \Re$ (20%).

1.2. Complete the sentence: A, B, C are mutually independent if and only if

Answer:

$P(A \cap B) = P(A)P(B), P(B \cap C) = P(B)P(C), P(A \cap C) = P(A)P(C),$
$P(A \cap B \cap C) = P(A)P(B)P(C).$

1.3. Bayes's Rule. Assume that $\{A_1, \ldots, A_n\}$ form a partition of Ω and that $p_m = P(A_m)$ and $q_m = P[B|A_m]$ for $m = 1, \ldots, n$. Derive $P[A_m|B]$ in terms of p's and q's.

Answer:

$$\begin{aligned} P[A_m|B] &= \frac{P(A_m \cap B)}{P(B)} = \frac{P(A_m)P[B|A_m]}{\sum_{k=1}^n P(A_k \cap B)} \\ &= \frac{P(A_m)P[B|A_m]}{\sum_{k=1}^n P(A_k)P[B|A_k]} = \frac{p_m q_m}{\sum_{k=1}^n p_k q_k}. \end{aligned}$$

1.4. Assume that X is equal to 2 with probability 0.4 and is uniformly distributed in $[0, 1]$ otherwise. Calculate $E(X)$ and $\text{var}(X)$. (Hint: Recall that $\text{var}(X) = E(X^2) - E(X)^2$.)

Answer:

$$E(X) = 0.4 \times 2 + 0.6 \times 0.5 = 1.1.$$

Also,

$$\begin{aligned} E(X^2) &= 0.4 \times 2^2 + 0.6 \times \int_0^1 x^2 dx = 0.4 \times 4 + 0.6 \times \frac{1}{3} \\ &= 1.8. \end{aligned}$$

Hence,

$$\text{var}(X) = 1.8 - (1.1)^2 = 0.59.$$

1.5. Two random variables X, Y are related so that $aX + Y = b$ for some real constants a and b. Given $E(X) = \mu, \text{var}(X) = \sigma^2$, express $E(Y)$ and $\text{var}(Y)$ in terms of μ and σ.

Answer:

$Y = b - aX$. Therefore:

$$\begin{aligned} E[Y] =& E[b - aX] \\ =& b - aE[X] \\ =& b - a\mu \end{aligned}$$

$$\begin{aligned} \text{var}(Y) =& E[(Y - E[Y])^2] \\ =& E[(b - aX - b + aE[X])^2] \\ =& E[a^2(X - E[X])^2] \\ =& a^2\text{var}(X) \\ =& a^2\sigma^2 \end{aligned}$$

2. Posterior Probability (10%)

Two manufacturing plants produce similar parts. Plant 1 produces 1000 parts, 100 of which are defective. Plant 2 produce 2000 parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from plant 1?

Answer:

Define the following events:

- A = the part is selected from Plant 1
- D = the part is defective

We wish to find $P(A \mid D)$. The problem statement gives us $P(A) = \frac{1}{3}$, $P(D \mid A) = \frac{1}{10}$, and $P(D \mid A^c) = \frac{3}{40}$. Bayes' rule:

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D \mid A)P(A)}{P(D)}$$

Since $\{A, A^c\}$ partition Ω, we can use the total probability theorem to find $P(D)$:

$$\begin{aligned} P(D) =& P(D \mid A)P(A) + P(D \mid A^c)P(A^c) \\ =& \frac{1}{10} \cdot \frac{1}{3} + \frac{3}{40}\left(1 - \frac{1}{3}\right) = \frac{1}{12} \end{aligned}$$

$$P(A \mid D) = \frac{\frac{1}{10} \cdot \frac{1}{3}}{\frac{1}{12}} = \frac{2}{5}$$

3. Posterior Probability (10%)

A number is selected at random from 1,2,...,100. Given that the number selected is divisible by 2, what is the probability that the number is divisible by 3 or 5?

Answer:

Define the following events:

- B = the number is divisible by 2
- C = the number is divisible by 3
- E = the number is divisible by 5

$$\begin{aligned} P(C \cup E \mid B) =& \frac{P((C \cup E) \cap B)}{P(B)} \\ =& \frac{P(C \cap B) + P(E \cap B) - P(C \cap E \cap B)}{P(B)} \\ =& \frac{\frac{16}{100} + \frac{10}{100} - \frac{3}{100}}{\frac{50}{100}} \\ =& \frac{23}{50} \end{aligned}$$

4. Expectation (15%)

There is a series of mutually independent Bernoulli experiments that individually have probability p of success and probability $(1-p)$ of failure. These experiments are conducted until the r^{th} success. Let X be the number of failures that occur until this r^{th} success. The pmf of X is:

$$p_X(k) = \binom{k+r-1}{k} p^r (1-p)^k, k \geq 0$$

a) Justify the pmf.

Answer:

$X = k$ if k failures and $r-1$ successes occur in any order, and then the next experiment is a success. There are $\binom{k+r-1}{k}$ possible orderings where this occurs, and each permutation has probability $p^r(1-p)^k$. Therefore:

$$p_X(k) = P(X = k) = \binom{k+r-1}{k} p^r (1-p)^k$$

b) Express $E(X)$ in terms of p and r.

Answer:

Using the definition of expectation:

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} k p_X(k) \\ &= \sum_{k=0}^{\infty} k \frac{(k+r-1)!}{k!(r-1)!} p^r (1-p)^k \end{aligned}$$

Noticing that k will cancel part of $k!$, we relate this to $p_X(k-1; r+1)$, that is the probability that $X = k-1$ in a similar experiment to $r+1$ successes.

$$\begin{aligned} E[X] =& \sum_{k=0}^{\infty} \frac{(k+r-1)!}{(k-1)!r!} p^{r+1} (1-p)^{k-1} \cdot r \frac{1-p}{p} \\ =& r \frac{1-p}{p} \sum_{k=0}^{\infty} p_X(k-1; r+1) \\ =& r \frac{1-p}{p} \end{aligned}$$

5. Independence (15%)

Show that if three events A, B, and C are mutually independent, then A and $B \cup C$ are independent.

Answer:

Events A and $B \cup C$ are independent if and only if $P(A \cap (B \cup C)) = P(A)P(B \cup C)$.

$$\begin{aligned} P(B \cup C) =& P(B) + P(C) - P(B \cap C) \\ =& P(B) + P(C) - P(B)P(C) \end{aligned}$$

because B and C are independent.

$$\begin{aligned} P(A)P(B \cup C) =& P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) \\ =& P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\ =& P((A \cap B) \cup (A \cap C)) \\ =& P(A \cap (B \cup C)) \end{aligned}$$

6. Probability Distribution (10%)

Find the allowable range of values for constants a and b such that the following function is a valid CDF

$$F(x) = 1 - ae^{-x/b} \text{ if } x \geq 0, \text{ and 0 otherwise.}$$

For those values of a and b, compute $P(-2 < X < 10)$ where X is the associated random variable.

Answer:

If $a > 1$, then $F(0^+) < 0$, which would be invalid. If $a < 0$, then $F(x) > 1$ for $x \geq 0$, which is also invalid. For $\lim_{x\to\infty} F(x) = 1$, we need $b > 0$. Therefore $0 \leq a \leq 1$ and $b > 0$.

Since $F(x) = P(X < x)$:

$$\begin{aligned} P(-2 < X < 10) =& P(X < 10) - P(X < -2) \\ =& F(10) - F(-2) \\ =& 1 - ae^{-10/b} - 0 \\ =& 1 - ae^{-10/b} \end{aligned}$$

7. Preemptive Maintenance (20%)

Assume, for simplicity, that the lifetime of a machine is uniformly distributed in $[0,1]$. If the machine fails, you face a cost equal to C, which includes replacing the machine with a new one. Replacing the machine before it fails with a new one costs $K < C$. You decide to replace the machine after $T < 1$ time units or when it fails, whichever comes first.

(a) Let X be the random time when you replace the machine. Calculate $E(X)$ in terms of T.

Answer:

We replace the machine after either it fails or it reaches age $T < 1$. Let W be the time when the machine fails. If $W < T$, then we face a cost of C after W seconds, whereas if $W > T$, we face a cost K after T seconds. Consequently, we face an average time equal to

$$E(\min\{T, W\}) = \int_0^1 \min\{T, w\} dw = \int_0^T w dw + \int_T^1 T dw = \frac{1}{2}T^2 + T(1-T) = T - 0.5T^2.$$

(b) Let Y be the random cost when you replace the machine (either K or C). Calculate $E(Y)$ in terms of T.

Answer:

We face a cost

$$E[Y] = C \cdot P(W < T) + K \cdot P(W \geq T) = CT + K(1-T)$$

(c) The average replacement cost per unit of time is $E(Y)/E(X)$. Find the value of T that minimizes that average cost.

Answer:

The average cost per unit of time is equal to

$$\frac{CT + K(1-T)}{T - 0.5T^2} = \frac{K + (C-K)T}{T - 0.5T^2}.$$

To minimize that cost, we set the derivative with respect to T equal to zero. We find

$$[K + (C-K)T](1-T) = (T - 0.5T^2)(C-K),$$

so that

$$K + (C-2K)T - (C-K)T^2 = (C-K)T - 0.5(C-K)T^2,$$

i.e.,

$$K - KT - 0.5(C-K)T^2 = 0,$$

so that

$$T = \frac{-K + \sqrt{K^2 + 2K(C-K)}}{C-K} = \frac{-K + \sqrt{2KC - K^2}}{C-K}.$$