Midterm 2, 10/29.

(1) $I_C = \int_C (ydx - xdy)$ where C is a segment of a straight line starting at $(0,0)$ and ending at $(1,1)$, i.e. it can be described as $C = \{(x(t), y(t)) = (t,t), 0 \le t \le 1\}$.

(a) (4 pnts) True or False? $I_C = 0$

$$\int_C (ydx - xdy) = \int_0^1 (t\,dt - t\,dt) = 0 \qquad T$$

(b) (4 pnts) True or False? The integral does not depend on continuous deformations of C with fixed endpoints.

$$\int_C (Pdx + Qdy) \text{ does not depend on } C \text{ if } \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad F$$

(2) (10 pnts) Find $u(x,y)$ such that the function $f(x,y) = u(x,y) + i\sin(x)\sinh(y)$ is analytic.

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}(\sin x\, \mathrm{sh}\, y) = \sin x\, \mathrm{ch}\, y, \quad \frac{\partial u}{\partial y} = -\frac{\partial}{\partial x}(\sin x\, \mathrm{sh}\, y) = -\cos x\, \mathrm{sh}\, y$$

$$u = -\cos x\, \mathrm{ch}\, y + C$$

(3) (10 pnts) Find Laurent series centered at $z = 0$ for the function $f(z) = \frac{6-2z}{(1-z)(z+3)}$ in the ring $1 < |z| < 3$

$$f = \frac{6-2z}{(1-z)(z+3)} = \frac{1}{1-z} + \frac{3}{z+3} = -\frac{1}{z(1-\frac{1}{z})} + \frac{1}{1+\frac{z}{3}} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n,$$

(4) $f(z) = z\cot(z)$

(a) (4 pnt) True or False? $Res_{z=0} f(z) = 1$

$$f = \frac{\cos z}{z \sin z} \to \frac{1 - \frac{z^2}{2} + \cdots}{z(z - \frac{z^3}{6} + \cdots)} = \frac{1}{z^2}\left(1 - \frac{z^2}{2} + \cdots\right)\left(1 + \frac{z^2}{6} + \cdots\right) = \frac{1}{z^2} + C + \ldots$$

F, $Res = 0$

(b) (4 pnt) True or False? $Res_{z=\pi} f(z) = \pi$

$$f = \frac{\cos z}{z \sin z}, \qquad F$$

T

(5) (4 pnt) True or False? $\int_{|z|=100} \frac{dz}{z(z-1)(z-2)} = 0$

$$\int_{|z|=100} \frac{dz}{z(z-1)(z-2)} = 2\pi i \underbrace{\operatorname{Res}_{z=\infty} f(z)}_{=0}, \quad f(z) = \frac{1}{z^3} + \cdots \text{ as } z \to \infty$$

(6) (10 pnts) Compute the integral $\int_0^\infty \frac{dx}{(x^2+1)(x^2+9)}$

$$I = \int_0^\infty \frac{dx}{(x^2+1)(x^2+9)} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+1)(x^2+9)} =$$

$$= \frac{2\pi i}{2}\Big(\underbrace{\operatorname{Res}_{x=i} \frac{1}{(x^2+1)(x^2+9)}} + \underbrace{\operatorname{Res}_{x=3i} \frac{1}{(x^2+1)(x^2+9)}} \Big)$$

$$\frac{1}{(x+i)(x^2+9)} \cdot \frac{1}{x-i},$$

$$\operatorname{Res}_{x=i} = \frac{1}{(x+i)(x^2+9)}\Big|_{x=i} = \frac{1}{2i \cdot 8}$$

$$\operatorname{Res} = \frac{1}{(x+i3)(x^2+1)}\Big|_{x=3i} = \frac{1}{6i \cdot (-8)}$$

$$= \pi i \left(\frac{1}{i16} - \frac{1}{i16 \cdot 4} \right) = \pi \frac{4-1}{16 \cdot 4} = \frac{3\pi}{64}$$