Physics 7B
Fall 2020
Lectures 2 & 3 Midterm #2 Solutions

Problem 1

a. Potential of a sphere is $-\frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ **1 formula + 1 sign**

Potential difference is $V = V_b - V_a = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$ **1 formula + 1 sign**

$$C = \frac{Q}{V} = \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}}$$

1

b.

$$E = \frac{1}{2}CV^2 = \frac{2\pi\epsilon_0 V^2}{\frac{1}{a} - \frac{1}{b}}$$

3

c.

$$Q = CV = \frac{4\pi\epsilon_0 V}{\frac{1}{a} - \frac{1}{b}}$$

3

$$\text{surface charge density} = \frac{Q}{A} = \frac{Q}{4\pi a^2} = \frac{\epsilon_0 V}{a^2\left(\frac{1}{a} - \frac{1}{b}\right)}$$

3

d. By Ohm's Law

$$\mathbf{J} = \sigma\mathbf{E} = \frac{\sigma Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

2 formula + 1 substitution + 1 vectors

$$I = \int \mathbf{J} \cdot \mathrm{d}\mathbf{A} = \frac{\sigma Q}{4\pi\epsilon_0 r^2} \cdot 4\pi r^2 = \frac{\sigma Q}{\epsilon_0}$$

1 formula + 1 substitution

Problem 2

A charge is distributed on a ring of radius R such that the ring has linear density charge density λ. If $\lambda = +A_0 \cos^2\theta$, find:

(a) The electric potential at a point a distance d_0 from the center of the ring (see figure below). If a charge $+q$ is placed at that point, find the electric potential energy of the charge.

Solution: (9 pts.) We can obtain the electric potential V a point a distance d_0 from the center by integrating over infinitesimal segments of the ring. The first step is to find the charge dq within an infinitesimal angle dθ,

$$\mathrm{d}q = \lambda R \mathrm{d}\theta = A_0 \cos^2\theta R \mathrm{d}\theta \qquad \text{(3pts.)} \tag{1}$$

Next, we set up the integral,

$$V = \int_0^{2\pi} \frac{k\mathrm{d}q}{\mathscr{r}} = \int_0^{2\pi} \frac{kA_0 \cos^2\theta R \mathrm{d}\theta}{\sqrt{d_0^2 + R^2}} \qquad \text{(3pts.)} \tag{2}$$

Finally, we evaluate the integral,

$$V = \frac{kA_0 R}{\sqrt{d_0^2 + R^2}} \int_0^{2\pi} \cos^2\theta \mathrm{d}\theta = \frac{\pi k A_0 R}{\sqrt{d_0^2 + R^2}} \qquad \text{(2pts.)} \tag{3}$$

The electric potential energy of the charge is then given by

$$U = qV = \frac{\pi k q A_0 R}{\sqrt{d_0^2 + R^2}} \qquad \text{(1pts.)} \tag{4}$$

(b) The direction and magnitude of the force that the ring exerts on a charge $+q$ at a distance d_0 from the center of the ring (see figure below).

Solution: (6 pts.) We first need to determine the direction of the electric field at a distance d_0 from the center. If we call the axis of the ring the x-axis, the electric field generated by an infinitesimal segment at θ and its opposite segment at $\theta + \pi$ cancels in the yz-plane because they have same magnitude and opposite direction in the yz-plane,

$$\cos\theta = -\cos(\theta + \pi) \implies \mathrm{d}q(\theta) \propto \cos^2\theta = \cos^2(\theta + \pi) \reflectbox{$\propto$} \mathrm{d}q(\theta + \pi) \qquad \text{(2pts.)} \tag{5}$$

(Need to say $\mathrm{d}q(\theta) = \mathrm{d}q(\theta+\pi)$ explicitly to get full credits. Otherwise, if $\lambda = A_0 \cos\theta$, it will look more like Problem 23.33 from the textbook.)

Thus, we can find the electric field by taking derivative of the potential,

$$\mathbf{E} = -\frac{\mathrm{d}V}{\mathrm{d}d_0}\hat{\mathbf{x}} = \frac{\pi k A_0 R d_0}{(d_0^2 + R^2)^{3/2}}\hat{\mathbf{x}} \qquad \text{(3pts.)} \tag{6}$$

Then we can find the force as

$$\mathbf{F} = q\mathbf{E} = \frac{\pi k q A_0 R d_0}{(d_0^2 + R^2)^{3/2}}\hat{\mathbf{x}} \qquad \text{(1pts.)} \tag{7}$$

If the charge q has initial velocity v_0 moving toward the center of the ring, find:

(c) the minimum value of v_0 so that the charge can exit on the other side of the ring.

Solution: (5 pts.) The easiest way to solve this problem is to use energy conservation. If the charge still has nonzero kinetic energy when it arrives at the center of the ring, it will exit on the other side. The initial kinetic energy it has is

$$K = \frac{1}{2}mv_0^2 \qquad \text{(2pts.)} \tag{8}$$

The electric potential energy it has to overcome to reach the center of the ring is

$$\Delta U = U(d_0 = 0) - U(d_0) = \pi k q A_0 \left(1 - \frac{R}{\sqrt{d_0^2 + R^2}}\right) \qquad \text{(2pts.)} \tag{9}$$

Finally, we can find v_0 by setting $K = \Delta U$,

$$\frac{1}{2}mv_0^2 = \pi k q A_0 \left(1 - \frac{R}{\sqrt{d_0^2 + R^2}}\right) \implies v_0 = \sqrt{\frac{2\pi k q A_0}{m}\left(1 - \frac{R}{\sqrt{d_0^2 + R^2}}\right)} \qquad \text{(1pts.)} \tag{10}$$

Problem 3

Consider an infinite cylinder with radius R made of an insulating material. A sphere of radius R/2 is scooped out of the cylinder. The center of the sphere lies on the axis of the cylinder (see figure below). The cylinder has a uniform charge density ρ.
Solution:

Find the electric field as a function of distance from the cylinder's main axis in the plane for

(i) **Points inside the sphere**

(ii) **Points outside the sphere and inside the cylinder.**

(iii) **Points outside the cylinder.**

The key is that we should consider the superposition of the electric fields from the infinite insulating cylinder with radius R with volume charge density ρ and an insulating sphere with radius $R/2$ and volume charge density $-\rho$. We first calculate the electric field for an insulating cylinder/sphere with generic radius R and volume charge density, then plug in the appropriate values. We will do this through Gauss's Law,

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

For an insulating sphere, using Gauss's Law, we consider a concentric Gaussian sphere of radius r. By symmetry $\vec{E}(\vec{r}) = |E(r)|\hat{r}$ and thus $\vec{E} \cdot \mathrm{d}\vec{A} = |E(r)|\,\mathrm{d}A$ as the differential area vector is also radial, $\mathrm{d}\vec{A} = \mathrm{d}A\hat{r}$. Then, the left hand side of the above equation simplifies to,

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = |E(r)| \oint \mathrm{d}A = |E(r)|4\pi r^2$$

The right hand side of Gauss's Law changes based on if $r < R$ or $r > R$.

$$\frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{enc} \mathrm{d}q = \frac{1}{\epsilon_0}\int_{enc} \rho\,\mathrm{d}V = \frac{1}{\epsilon_0}\rho V_{enc} = \begin{cases} \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r^3 & r < R \\ \frac{\rho}{\epsilon_0}\frac{4}{3}\pi R^3 & r > R \end{cases}$$

Equating both sides, we find that,

$$\vec{E}_{sphere}(r) = \begin{cases} \frac{\rho}{3\epsilon_0} r & r < R \\ \frac{\rho\pi}{3\epsilon_0}\frac{R^3}{r^2} & r > R \end{cases}$$

For an insulating cylinder, using Gauss's Law, we consider a concentric Gaussian cylinder of azimuthal radius[1] r and length l. By symmetry $\vec{E}(\vec{r}) = |E(r)|\hat{r}$ and thus $\vec{E} \cdot \mathrm{d}\vec{A} = |E(r)|\,\mathrm{d}A$ as the differential area vector is also azimuthal radial, $\mathrm{d}\vec{A} = \mathrm{d}A\hat{r}$. Notice that this is only true on the curved edge of the cylinder, and thus the surface integral in Gauss's Law will not have any contributions from the flat ends of the cylinder. The left hand side of the above equation simplifies to,

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = |E(r)| \oint \mathrm{d}A = |E(r)|2\pi r l$$

The right hand side of Gauss's Law changes based on if $r < R$ or $r > R$.

$$\frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\int_{enc} \mathrm{d}q = \frac{1}{\epsilon_0}\int_{enc} \rho\,\mathrm{d}V = \frac{1}{\epsilon_0}\rho V_{enc} = \begin{cases} \frac{\rho}{\epsilon_0}\pi r^2 l & r < R \\ \frac{\rho}{\epsilon_0}\pi R^2 l & r > R \end{cases}$$

[1] By this, I mean the radius variable in cylindrical coordinates.

Equating both sides, we find that,

$$\vec{E}_{cylinder}(r) = \begin{cases} \frac{\rho}{2\epsilon_0} r & r < R \\ \frac{\rho}{2\epsilon_0} \frac{R^2}{r} & r > R \end{cases}$$

In our specific case, the cylinder has charge density ρ and radius R while the sphere has charge density $-\rho$ and radius R. Notice that the azimuthal radius in cylindrical coordinates and polar radius in spherical coordinates coincides when we consider points on the xy plane.

$$\vec{E}(d) = \vec{E}_{cylinder} + \vec{E}_{sphere} = \begin{cases} \frac{\rho}{\epsilon_0} \left(\frac{d}{2} - \frac{d}{3}\right) & r < R/2 \\ \frac{\rho}{\epsilon_0} \left(\frac{d}{2} - \frac{1}{3}\frac{1}{d^2}\left(\frac{R}{2}\right)^3\right) & R/2 < r < R \\ \frac{\rho}{\epsilon_0} \left(\frac{R^2}{2d} - \frac{1}{3}\frac{1}{d^2}\left(\frac{R}{2}\right)^3\right) & r > R \end{cases}$$

Thus,

$$\boxed{\vec{E}(d) = \begin{cases} \frac{\rho r}{6\epsilon_0} \hat{r} & d < R/2 \\ \frac{\rho}{2\epsilon_0} \left(d - \frac{1}{12}\frac{R^3}{d^2}\right) \hat{r} & R/2 < r < R \\ \frac{\rho}{2\epsilon_0} \frac{R^2}{d} \left(1 - \frac{1}{12}\frac{R}{d}\right) \hat{r} & r > R \end{cases}}$$

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Grading Rubric

Out of 20 points:

- + 8 E field of Cylinder
  - + 3 Gauss's Law $\oint \vec{E} \cdot \mathrm{d}A$
  - + 3 Gauss's Law Q_{enc}
  - + 2 Correct Answer
- + 8 E field of Sphere
  - + 3 Gauss's Law $\oint \vec{E} \cdot \mathrm{d}A$
  - + 3 Gauss's Law Q_{enc}
  - + 2 Correct Answer
- + 4: Method
  - + 2 Superposition Principle
  - + 2 Correct Answer

-1 Off by a sign

Problem 4

4. (a) To determine the field at the point of contact, we use the principle of superposition. Using Gauss' law, we obtain the following expressions for the fields due to the positive and negative spheres:

$$\mathbf{E}_{+} = \frac{\frac{4\pi}{3}\rho r^3}{4\pi\epsilon_0 r^2}\hat{x} = \frac{\rho r}{3\epsilon_0}\hat{x}, \tag{1}$$

$$\mathbf{E}_{-} = \frac{-\frac{4\pi}{3}\rho s^3}{4\pi\epsilon_0 s^2}(-\hat{x}) = \frac{\rho s}{3\epsilon_0}\hat{x}. \tag{2}$$

At the origin $r = s = R$, so we obtain

$$\mathbf{E}(x = 0) = \frac{2\rho R}{3\epsilon_0}\hat{x} \tag{3}$$

(b) At the center of the right sphere (the negative sphere), $s = 0$. Therefore, the negative sphere does not make a contribution to the total electric field at this point. The electric field due to the positive sphere is given by Gauss' law:

$$\mathbf{E}_{+} = \frac{\frac{4\pi}{3}\rho R^3}{4\pi\epsilon_0 r^2}\hat{x} = \frac{\rho R^3}{3\epsilon_0 r^2}\hat{x} \tag{4}$$

At the center of the right sphere, $r = 2R$, so we obtain

$$\mathbf{E}(x = R) = \frac{\rho R}{12\epsilon_0}\hat{x} \tag{5}$$

(c) Because the charge is uniformly distributed and because the two spheres create an electric field external to their volumes that is the same as the electric field due to a point charge, we may treat these two spheres as point charges. Thus, the dipole moment is

$$\mathbf{p} = \rho\frac{4\pi R^3}{3}(2R)(-\hat{x}) = \frac{-8\pi\rho R^4}{3}\hat{x}. \tag{6}$$

Midterm 2, Problem 5 Rubric: Lectures 2&3:

Part #	Point Total	(9-10)/10	(7-8)/10	(5-6)/10	(3-4)/10	(1-2)/10	0/10
Part A	10	Completely Correct	1 minor mistake, overall conceptual understanding is there.	More than 1 mistake.	Largely incorrect, but partial understanding displayed.	Complete lack of conceptual understanding.	No answer.
Part B	10	Completely Correct	1 minor mistake, overall conceptual understanding is there.	More than 1 mistake.	Largely incorrect, but partial understanding displayed.	Complete lack of conceptual understanding.	No answer.

<u>Solution:</u>

5a).

By the principle of continuity, the same current enters and exits the cone.

5b).

Solving for the areas at each end of the cone:

Left Hand Side Area = $\frac{15*\pi*a^2}{64}$

Right Hand Side Area = $\frac{17*\pi*a\text{^}2}{81}$

Thus RHS Area < LHS Area

This means that the current density at the RHS is greater, resulting in a higher electric field at the RHS.