INDENG 172: Probability and Risk Analysis for Engineers

Fall 2020 - Final

Due by: 12/16/2020 at 11:00 AM PST

Name: ______________________ Student ID: ____________

Instructions

This exam is open book, open notes, and open resources. However, your work must be written/typed by you and be your work alone.

The exam consists of two parts, 10 'Standard Problems' (of which you may choose 6) and 2 'Create Your Own Question and Solution' (which have been pre-released).

Your solutions must be uploaded to Gradescope by 12/16/2020 at 11:00 AM PST. Late submissions without prior approval will not be accepted. If you have any questions or issues please send an email to the course staff with subject line "IEOR 172 FINAL QUESTION".

Standard Problems

Directions

Select 6 problems to solve, if you submit more than 6 solutions, your **lowest** graded solutions will be counted. All questions are worth the same amount of points, so you are encouraged to tackle the problems you find the easiest.

To earn full credit it is not sufficient to merely have the correct numerical answer, you must also show your work, and explain your reasoning to demonstrate that you understand the concepts being tested. Partial credit will be awarded.

Problem 1

You and your partner are getting ready to go out for dinner. You wear each other's socks cuz you're cute like that. Reaching into your sock drawer, containing six individual black socks and six individual white socks, you randomly pull out four of them and decide which two socks you each will wear.

1. What is the probability that using the four socks you grabbed you can each get dressed and have a proper pair of socks?

Later, at dinner the menu has a choice of three appetizers, two entrees, and four desserts. How many different combinations of meals can you have if

2. Your partner chooses the cheapest appetizer for you, and you choose an entree and a dessert?
3. You choose an appetizer and an entree, but may decide to skip dessert.

Problem 2

An urn has 1 red ball, 2 blue balls, and 1 green ball. A ball will be selected at random. If it is not red, you stop. If it is red, it is put back in the urn and a ball is again selected at random, and then you stop. So in total, you select one or two balls. Let X be the number of red balls you selected, and let Y be the total number of blue balls selected.

1. Give a table showing the joint probability mass function of X and Y, $p_{X,Y}(x,y)$. You may use the table below as a starting point:

x	y=0	y=1

2. Give the conditional probability mass function of Y, given X = 0.
3. Are X and Y independent? Make sure you justify your answer mathematically.

Problem 3

You're supposed to be watching some pre-recorded lecture, but find your attention wandering. (It definitely be like that). You pick up your phone and start scrolling Instagram. You tell yourself you'll stop when you see the "You're All Caught Up" banner at the bottom. Sometimes you get captured by the ads and spend some time dreaming about the products before returning to the top of your feed and starting to scroll all over again. To simplify the problem, let's assume your feed (including the ads) are fixed and there are only 3 specific events that could happen while scrolling:

1. With probability .5 you get lost in a Shein ad for 12 minutes before returning to the top of your feed to scroll through things again.
2. With probability .3 you get lost in a Disney+ ad for 7 minutes before returning to the top of your feed to scroll through things again.
3. With probability .2 you scroll for 5 minutes and reached the banner at the bottom of your feed.

What is the expected number of minutes until you finally get off Instagram and turn your attention back to the lecture?

Problem 4

I made three batches of 15 cookies each and combined them all into one big cookie jar. My first batch consisted of a 9 chocolate chip cookies, 4 snickerdoodles, and 2 oatmeal raisin cookies. My second batch consisted of 3 chocolate chip cookies, 8 snickerdoodles, and 4 oatmeal raisin cookies. My third batch consisted of 7 chocolate chip cookies, 3 snickerdoodles, and 5 oatmeal raisin cookies.

Consider the following 3 scenarios (each scenario is unrelated to the others)

1. I want three cookies and take them out of the jar. What is the probability that all three are chocolate chip cookies?
2. I pick a cookie from the jar and it's a chocolate chip cookie. What is the probability that I baked it in my first batch?
3. I hate snickerdoodles, don't want them, and don't think they deserve to exist. I will take cookies from the jar one at a time. If I get a

snickerdoodle, I will throw it in the trash and pick another cookie. If I get a chocolate chip or oatmeal raisin cookie, I will put it aside to eat and continue until I have put aside 3 cookies to eat. What is the expected number of snickerdoodles that will be left in the jar when I'm done? (Hint: Although you *can* derive this from scratch, there is a known (but perhaps less common) discrete distribution that can be applied here)

Problem 5

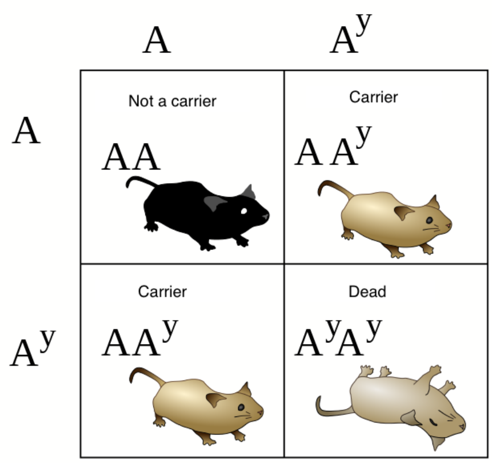

Figure 1: Original source Wikipedia user Jcfidy

For this problem we will consider a simplified version of lethal genes and inheritance in mice. There are two forms of a gene A and A^y. The gene A is the regular form, and A^y is a mutation that is lethal but recessive. Every mouse has a pair of genes that can be any combination of these genes. A mouse is called a "carrier" if only one of its genes contains the lethal gene A^y. If both of the mouse's genes are A^y it will not be viable and will die at birth. A viable mouse (carrier or not) will die at birth from other unrelated factors with probabilty $\frac{1}{2}$. We will assume that every child mouse inherits one gene from each of its parent's pair of genes (independently with equal likelihood) to form its own pair of genes. See Figure 1 for a visual explanation.

I have a very large collection of adult (viable) mice in my lab. You may assume that they are equally likely to have any of the viable gene pairs. I randomly select a male and female, separate them from the group, and observe their offspring. After 5 pregnancies, only 3 offspring have survived past birth.

1. What is the probability that the male and female are both carriers?
2. If I know that the male is a carrier, what is the probability that the female is one as well?
3. If I know that the male is not a carrier, what is the probability that the female is one?

Problem 6

Let X_i be a sequence of independent and identically distributed (i.i.d) continuous random variables. We say a peak occurs at time n if

$$X_n \geq \max\{X_1, ..., X_{n-1}\}.$$

Let N be the total number of peaks that have occurred by time n. Find $\mathrm{E}[N]$ and $\mathrm{Var}(N)$.

Problem 7

Suppose a newly established casino in Berkeley has the following game. You pay a \$5 entry fee and are given a well shuffled deck of cards. The deck consists of n red cards, m black cards, and k blank cards, with $n > m$. After paying your \$5 entry fee you draw a card. If you draw a red card you earn \$1, if you draw a black card you must pay \$1, if you draw a blank card, nothing happens. In order to be allowed to draw a second time, you must show the casino that you would be able to pay up if you get a black card next (i.e. show that you have at least \$1 in your pocket). Similarly for your third draw, fourth draw, and so on up until the $n + m + k$th draw.

Assume, you start out with \$5, and after paying the entry fee (and thus earning your first draw), you have \$0 in your pocket. As an example, if your draws were $[+1, 0, +1, -1, 0, -1]$ then you were only allowed to draw 6 cards, and had to stop after the 6th draw, because

$$
\begin{aligned}
&0 < 1 \\
&0 < 1 + 0 \\
&0 < 1 + 0 + 1 \\
&0 < 1 + 0 + 1 - 1 \\
&0 < 1 + 0 + 1 - 1 + 0 \\
&0 = 1 + 0 + 1 - 1 + 0 - 1
\end{aligned}
$$

What is the probability that you will be able to draw **all** of the cards in the deck without being forced to stop because you have $0 in your pocket?

Problem 8

Pfoser is testing two new vaccines for COVID-19 on volunteer participants. Vaccine B has a higher success rate than Vaccine A for female participants as well as for the male participants.

1. Prove or disprove the following statement: Vaccine B has a higher overall success rate on all participants. (Note: to disprove a statement, you need to find a counterexample.)

If $\mathrm{E}[Y|X] = 1$, prove or disprove the following two statements:

2. $\mathrm{E}[XY] = \mathrm{E}[X]$
3. $\mathrm{Var}[XY] \geq \mathrm{Var}[X]$

Problem 9

We are examining the vehicles that cross the Bay Bridge. The weight of a vehicle, the number of passengers, and the weight of individual passengers are all random with the following properties:

- Vehicle weight (without passengers) has mean 3,500 lbs and standard deviation 1,000 lbs.
- The number of passengers in a vehicle has mean 2.5 and standard deviation 1.2.

- The weight of each passenger has mean 140 lbs and standard deviation 100 pounds.

You may assume each of these random variables are independent of each other.

1. Let W be the total weight of a vehicle crossing the Bay Bridge, including passengers. Find the mean and standard deviation of W.

2. There is a new proposed fee schedule in which each vehicle crossing the bridge is charged according to its total weight. Let $F = \alpha W + \beta$, where $\alpha = 0.002$ and $\beta = 1$ be the fee payed by a crossing vehicle. Find the mean and standard deviation of F.

3. During the most recent Black Friday, Bay Bridge's vehicle counter registered 5,000 vehicle crossings with a total toll receipt of \$43,260. Find or approximate the probability that an arbitrary vehicle didn't pay the toll. (Hint: You may view the total toll receipts as a sum of 5,000 i.i.d rvs)

Problem 10

Due to COVID-19, a new hand sanitizer factory is now established in Berkeley. Each bottle of hand sanitizer (150 grams) is intended to contain exactly 80% ethyl alcohol (120 grams). However, the manufacturing process is not perfect. Assume the error (in grams) of the amount of alcohol in a bottle of hand sanitizer is a normal random variable with mean zero and standard deviation of 5 grams.

1. Let W denote the amount of alcohol contained in a bottle of hand sanitizer produced by the factory, and let ϵ denote the error the machine makes. What is the PDF of W?

2. Let $X = (\epsilon/5)^2$. Compute the PDF of X; do not forget to specify the range of values X can take.

3. Compute the moment generating function of X,$M_X(t)$; do not forget to specify the range of values of t for which $M_X(t)$ is finite.

4. Use your computed $M_X(t)$ to compute the mean and variance of X.

Create Your Own Question and Solution

Directions For each of the following questions read the probability and application topics. You must then create your own question and solution related to the probability topic using the application topic. An example of this is given below.

The question and solution must be written by you and be uniquely yours. However, you may use the textbook, the reference material, the internet, or any other resource you want for inspiration. Make sure you cite your inspirational sources or declare that there are none. Failure to include such a statement will result in no credit. For more details see the grading information section below.

Question & Solution 1

Probability Topic: A continuous distribution and a concept covered in lecture after the midterm.

Application Topic: Anything you find interesting or fun.

Required components:

- Inspiration statement
- Question
- Solution

Question & Solution 2

Probability Topic: From Lectures on Limit Theorems (Part 1 and Part 2) or Markov Chains.

Application Topic: Choose one: Weather, Manufacturing, Data Analysis, Robotics, Prediction, AI Safety, Communication Systems, or Energy.

Required components:

- Inspiration statement

- Question
- Solution

Grading information

Standard Problem grading (similar to problem set grading, 25 point for each question)

You will be awarded points for providing the numerically correct solution, showing your work, and explaining your logic and thought process in a manner that demonstrates understanding of the basic concepts.

You are only required to answer 6 of the 10 problems. If you submit more than 6 solutions, your **lowest** graded solutions will be used. This encourages you to only submit solutions that you are confident about.

'Create Your Own Question and Solution' Grading (25 points each)

- Citing your inspiration or declaring that there is none is an important part of this question. No credit will be given if you fail to include a statement about inspiration.
- If you simply copy from a source, another student, or previous work of yours, you will earn 0 points. We are looking for changed details, numbers, etc.. Be creative!
- In addition, if your question is substantially similar to other students and you fail to cite your inspiration this may qualify as violating the honor code (read: Cheating) and the issue may be referred to the Center for Student Conduct.
- You will be awarded points for expressing a creative, clear, and unambiguous question of reasonably similar difficulty to the weekly problem sets, providing the numerically correct solution, showing your work, and explaining your logic and thought process in a manner that demonstrates understanding of the basic concepts. The exact details of the rubric used may differ from the one in the syllabus, but the overall idea is the same.

Question & Solution Example

Probability Topic: Expected value

Application Topic: Vaccines

Required components:

- Inspiration statement
- Question
- Solution

Inspiration: No inspiration, I have thought of this on my own.

Question: We would like to re-test the efficacy of 4 vaccines (including 1 placebo) for Fauxvid-19 in mice. Each of our 100 lab mice are randomly (and independently) given one of the four vaccines. We then expose them to Fauxvid-19 and check for active infection.

Previous research has indicated that if a mouse is given the placebo it will have an active infection with probability .9, if given Vaccine 1 it will have an active infection with probability .75, Vaccine 2 yields probability .51, and Vaccine 3 yields probability .3.

What is the expected number of mice in our lab that will have an active infection?

Solution: Let Y be a random variable denoting the number of infected mice. Let X_i be 1 if the ith mouse is infected and 0 otherwise. Then $Y = \sum_{i=1}^{100} X_i$.

$$\begin{aligned} E\left[Y\right] &= E\left[\sum_{i=1}^{100} X_i\right] \\ &= \sum_{i=1}^{100} E\left[X_i\right] \end{aligned}$$

This is by linear of expectations. Using the definition of expected value we get,

$$\begin{aligned} E[Y] &= \sum_{i=1}^{100} E[X_i] \\ &= \sum_{i=1}^{100} (1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)) \\ &= \sum_{i=1}^{100} (P(X_i = 1)) \end{aligned}$$

Since X_i is i.i.d. we can drop the i and focus on an arbitrary single mouse. Let $V_0, V_1, \ldots$ be the event that the mouse gets the placebo, vaccine 1, etc.. Then using the assumption in the question we have,

$$\begin{gathered} P(X = 1 \mid V_0) = .90 \\ P(X = 1 \mid V_1) = .75 \\ P(X = 1 \mid V_2) = .51 \\ P(X = 1 \mid V_3) = .30 \end{gathered}$$

Since all the V's are mutually exclusive and their union encompasses the whole sample space (exhaustive partition), we can use the Law of Total Probability to find $P(X = 1)$ as follows,

$$\begin{aligned} &P(X = 1) \\ &= P(X = 1 \mid V_0)P(V_0) + P(X = 1 \mid V_1)P(V_1) + P(X = 1|V_2)P(V_2) + P(X = 1 \mid V_3)P(V_3) + P(X = 1 \mid V_4)P(V_4) \\ &= .9\left(\frac{1}{4}\right) + .75\left(\frac{1}{4}\right) + .51\left(\frac{1}{4}\right) + .3\left(\frac{1}{4}\right) \\ &= 0.615 \end{aligned}$$

$P(V_0) = P(V_1) = P(V_2) = P(V_3) = P(V_4) = \frac{1}{4}$ because we assumed they are all equally likely. Thus,

$$\begin{aligned} E[Y] &= \sum_{i=1}^{100} (P(X = 1)) \\ &= 100(.615) \\ &= 61.5 \end{aligned}$$

So the expected number of mice in our lab that will have an active infection is 61.5