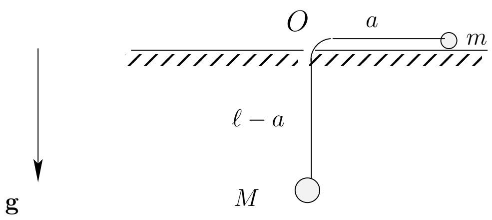
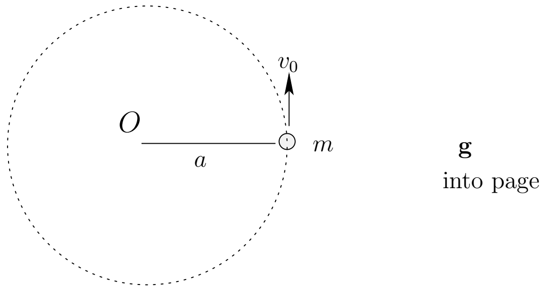
University of California, Berkeley
Mechanical Engineering
ME104 Engineering Mechanics **SOLUTIONS**
Test 2 S20 *Prof S. Morris*

1. (100) Mass m lies on a smooth horizontal table, and is connected by a string of constant length ℓ to the mass M hanging below the table. Initially, mass M is at rest; mass m is then distant a from the hole O, and has velocity v_0 perpendicular to the string.



(i) Side view of the initial state



(ii) Top view of the initial state

(a) What quantities are conserved during the subsequent motion of the system consisting of the two connected masses.?

(b) Using your answer to part (a), derive the expression giving $\dot{r}$, the time rate of change of the distance between the mass m and point O, in terms of r, a, g m, M and v_0.

<u>SOLUTION</u> (a) Because no external torque acts about the vertical axis through O, angular momentum about that axis is conserved; because the table is smooth, the sum of the kinetic energy of the two masses and the potential energy is conserved.

(b) Let the z–axis point downwards (parallel) to $\mathbf{g}$. Then, because the string has constant length ℓ, $z + r = \ell$ so that the velocity of M is equal and opposite to $v_r = \dot{r}$ the radial component of velocity of m.

Because angular momentum is conserved

$$mrv_\theta = mav_0. \tag{1.1a}$$

Because the total mechanical energy is conserved

$$\frac{1}{2}m\{v_r^2 + v_\theta^2\} + \frac{1}{2}Mv_r^2 - Mgz = \frac{1}{2}mv_0^2 - Mg(a - \ell).$$

Terms on the left side of this expression represent the kinetic energy of mass m, and the kinetic and potential energies of mass M. Terms on the right hand side represent the initial kinetic energy of m and the initial potential energy of M. The potential energy is negative in each case because the table top is taken to be the reference level, and M is below it.

By rearranging terms, and using $z + r = \ell$,

$$\frac{1}{2}\{m + M\}v_r^2 + \frac{1}{2}mv_\theta^2 - Mg(a - r) = \frac{1}{2}mv_0^2. \tag{1.1b}$$

By eliminating v_θ between (1.1a) and (1.1b), then using $v_r\dot{t}$,

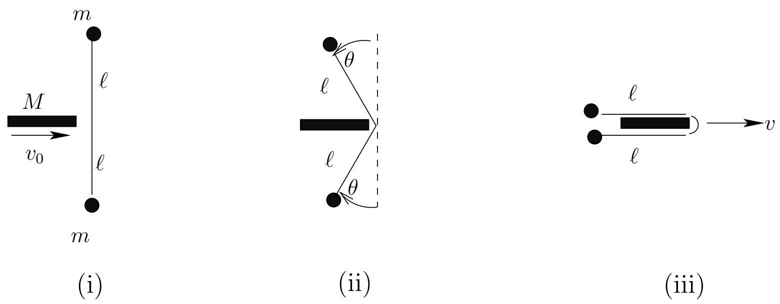
$$\frac{1}{2}\{m + M\}\dot{r}^2 + \frac{1}{2}mv_0^2\frac{a^2}{r^2} - Mg(a - r) = \frac{1}{2}mv_0^2. \tag{1.2}$$

Last, on multiplying by $2r^2/(m + M)$,

$$r^2\dot{r}^2 = \frac{2Mg}{M + m}(a - r)\Big\{r^2 - \frac{mv_0^2}{2Mg}(r + a)\Big\}. \tag{1.3}$$

This expresses $\dot{r}$ in terms of r, a, g, m, M and v_0.

2. (100) Two small spheres each of mass m are connected by a massless inextensible cord of length 2ℓ. The spheres are initially at rest on a smooth horizontal surface. The projectile of mass M strikes the cord in the middle with velocity v_0 in the x–direction.



(a) Immediately after the projectile strikes the cord, the x–component of velocity is zero for each sphere. Using an appropriate free–body diagram, explain why this must be so.

(b) Hence find the velocity of the projectile immediately after it strikes the cord.

(c) Because the cord is inextensible, the continuing motion deforms the cord, causing the spheres to move as shown in figure (ii). Ultimately, the two spheres approach each other, as in figure (iii). Determine the velocity v of the projectile at the instant immediately preceding the collision between two spheres.

(d) For that same instant, determine $\dot{\theta}$.

<u>SOLUTION</u> (a) The appropriate free–body diagram is that for one of the spheres (say the upper one in the diagram). It should show that when the projectile strikes the cord, the force exerted by the cord could only be in the y–direction (along the cord). Consequently, x–momentum is not imparted to m until the cord is bent into the V–shape shown in Fig.ii.

(b) Because no external forces act on the system consisting of the two spheres and the the projectile, its momentum is conserved. Because neither sphere has x–momentum immediately after the projectile strikes the cord, the velocity of the projectile itself remains equal to v_0.

(c) For the configuration shown in Fig.iii, all three particles have the smae x–componenent of velocity v. Because no external forces act on the system, the x–component, in particular, is conserved:

$$m_0 v_0 = (m_0 + 2m)v.$$

The velocity v of the projectile is therefore

$$v = \frac{m_0}{m_0 + 2m} v_0. \tag{2.1}$$

(d) Because no external forces perform work on the system, the kinetic energy does not change during the interval separating the collision of the projectile with the cord, and the collision of the two spheres:

$$\frac{1}{2} m_0 v_0^2 = \frac{1}{2} m_0 v^2 + 2 \times \frac{1}{2} m \{ v^2 + \ell^2 \dot{\theta}^2 \} \tag{2.2}$$

The ever–present idea of relative velocity has been used here:

$$\mathbf{v}_{\text{sphere}} = \mathbf{v}_{\text{proj.}} + \mathbf{v}_{\text{sphere/proj.}}$$

On the right hand side, $\mathbf{v}_{\text{proj.}} = v\mathbf{e}_x$ and, because the cord is inextensible and the spheres are about to collide, for the upper sphere in Fig.iii, $\mathbf{v}_{sphere/proj.} = -\ell\dot{\theta}\mathbf{e}_y$. Hence, for either sphere, $v_{\text{sphere}}^2 = v^2 + \ell^2\dot{\theta}^2$, as in Eq.(2.2).

By eliminating v between (2.1) and (2.2), then solving for $\dot{\theta}$,

$$\dot{\theta} = \frac{v_0}{\ell} \Big(\frac{m_0}{m_0 + 2m} \Big)^{1/2}. \tag{2.3}$$

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