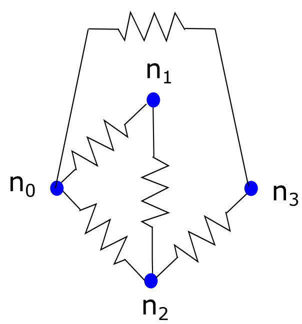
MIDTERM for MATH 170

Instructions: Your completed midterm must be submitted to the bCourse site for Math 170, by *10 AM on Thursday, October 22*. On Wednesday, October 21, after the exam is released, you will find a submission link called "Midterm" in your bCourse "Assignment" section. Submit your midterm via that link.

You are allowed to look at anything on the Math 170 bCourses site and the book by Franklin. However, *you may not consult with any person, nor read any other book, nor go to any other online source for help.*

Please write out your answers clearly, and explain every step of your reasoning. Good luck!

Problem 1: The resistors in the following circuit all have the same value $r > 0$. Assume the voltage at n_0 is set to 1 and the voltage at n_3 is set to 0.



Use the minimization principle explained in class to calculate the voltages at the nodes n_1, n_2.

Problem 2: (i) Show that the matrix

$$A = \begin{bmatrix} 4 & 0 & 7 \\ -2 & 1 & 0 \end{bmatrix}$$

does not have a saddle point.

(ii) Use linear programming to find a mixed strategy saddle point (p_0, q_0) for the two-person, zero-sum game with payoff matrix A. What is the value ω of this game?

Problem 3: Suppose $A = A^T$ is a symmetric $n \times n$ matrix. Assume $b \in \mathbb{R}^n$ and consider the linear programming problem:

$$\begin{cases} \text{minimize } b \cdot x, \text{ subject to} \\ Ax = b, \ x \geq 0. \end{cases}$$

Show that any feasible x is optimal.

Problem 4: Recall that the following primal problems are equivalent:

$$(\mathrm{P}) \begin{cases} \text{minimize } c \cdot x, \\ \text{subject to} \\ Ax = b, x \geq 0 \end{cases} \qquad (\mathrm{P}^*) \begin{cases} \text{mininimize } c \cdot x, \\ \text{subject to} \\ \tilde{A}x \geq \tilde{b}, x \geq 0, \end{cases}$$

for

$$\tilde{A} = \begin{bmatrix} A \\ -A \end{bmatrix}, \ \tilde{b} = \begin{bmatrix} b \\ -b \end{bmatrix}.$$

Write down the corresponding dual problems (D),(D*) and show that (D) has a feasible solution if and only if (D*) has a feasible solution.

Problem 5: We have n differently shaped containers sitting on a table, and the cross sectional area of the k-th container at height y above the table is $a^k(y) > 0$. We connect the containers with small tubes at their bases that allow water to flow freely between them, and then pour in a given volume $V > 0$ of water. Show that at equilibrium the heights of the water in each container are all equal.

(Hint: If x_k is the height of the water in the k-th container, the gravitational potential energy of the water in that container is $\int_0^{x_k} y a^k(y)\, dy$. The equilibrium water levels minimize the total potential energy, subject to the constraint that the total amount of water distributed among the containers is V.)