Math 185 First Midterm October 6, 2020

Name: ______________________________

- You have 80 minutes to complete the exam, 9:40-11:00am.
- Please write your name and page number on every page that you submit. The submission deadline is at 11:10am.
- This is a open-book exam, you can use your textbook and notes.
- You may only use the results covered in class so far, including results in the lecture note and results in Stein up to Chapter 2.
- If you have question during the exam, you may contact me in zoom,
- Please write neatly. Answers which are illegible for the reader cannot be given credit.

Good Luck!

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	20	
9	10	
Total	100	

1. (10 points, 2 points each)

   (1) Use z and $\overline{z}$ to express $\mathrm{Re}(z), \mathrm{Im}(z), |z|^2$.

   (2) If $z = 2020 + 1006i$, then $|z/\overline{z}| =?$.

   (3) If $z = (1/2)e^{i\pi/3}$, then $1/\overline{z} =?$

   (4) Give an example of a continuous function $f : \mathbb{C} \to \mathbb{C}$, where f is holomorphic at 0 but no other point in $\mathbb{C}$. (no justification needed)

   (5) State the Cauchy-Riemann criterion for a function f to be holomorphic.

2. (10 points, 2 points each) Let f be a holomorphic function on the unit open disk $\mathbb{D}$. Determine whether the following statements is true or false. No justification needed.

   (1) There exists a sequence of polynomials f_n, such that for any compact set $K \subset \mathbb{D}$, f_n converges to f uniformly on K

   (2) If f vanishes at infinitely many points in $\mathbb{D}$, then f is zero.

   (3) If there is a point $z_0 \in \mathbb{D}$, such that $f^{(n)}(z_0) = 0$ for all $n = 0, 1, 2 \cdots$, then $f = 0$.

   (4) Let γ be a closed piecewise smooth curve in $\mathbb{D}$, possibly with self-intersection, then it is possible that $\int_\gamma f(z)dz \neq 0$.

   (5) If $f(0) = 0$ and $f'(0) = 1$, then $f(z) = z$.

3. (10 points) Let $\Omega \subset \mathbb{C}$ be a region (open and connected subset), and $f : \Omega \to \mathbb{C}$ a holomorphic function. Suppose there is a line $L \subset \Omega$, such that f is constant on L. Show that f is constant in Ω.

4. (10 point) Show that the function $f : \mathbb{C} \backslash [0, 1] \to \mathbb{C}$

$$f(z) = \int_0^1 \frac{1}{z - t} dt$$

   is holomorphic in $\mathbb{C} \backslash [0, 1]$, and its derivative is

$$f'(z) = \int_0^1 \frac{-1}{(z - t)^2} dt.$$

   (Hint: Use difference quotient to compute the derivative. Do not pass differentiation under the integral sign without justification.)

5. (10 point) Let $K \subset \mathbb{C}$ be a compact set, and $f : K \to \mathbb{C}$ is a continous function. Is it always possible to find a sequence of polynomials $f_n(z)$, such that f_n converges to f uniformly on K? If yes, give a reference. If no, give your reason and a counter example.

6. (10 point) If $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, and there is a constant $C > 0$, such that $|f(z)| < C(1 + |z|)$. Show that $f(z) = a + bz$ for some $a, b \in \mathbb{C}$.

7. (10 point) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that there exists a point $z_0 \in \mathbb{C}$ and an open neighborhood $D_r(z_0)$, such that $f(\mathbb{C}) \cap D_r(z_0) = \emptyset$. Show that f is a constant function.

8. (20 points, 10 points each) Evaluate the following contour integrals.

(1)

$$\oint_{|z|=1} \frac{(z+2)(z+3)}{z(z+4)(z+5)} dz =?$$

(2) For any real number $a > 1$, evaluate

$$\oint_{|z|=1} \frac{1}{|z-a|^2} |dz|$$

9. (10 point) Let $g(z)$ be a holomorphic function on an open neighborhood of $\overline{\mathbb{D}}$, $z_0 \in \mathbb{C}$ with $|z_0| = 1$. Let $f(z) = \frac{g(z)}{z-z_0}$. Consider the follower power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{D}.$$

Show that

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

(Hint: write $g(z) = g(z_0) + (z - z_0)h(z)$.)