**Midterm Exam 2 PDF version**
Instructor: Prof. Raja Sengupta

April 15, 2020
25 questions, 100 points, 50 minutes
16 pages

**Name:** ____________________________

**Student ID:** ____________________________

**Statement of Academic Integrity**
UC Berkeley Honor Code: "As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."
On my honor, I will neither give nor receive any assistance in taking this exam. I will not use any program other than MATLAB on my computer and have turned off all my internet connections.

Signed: ____________________________________________

**Instructions**

1. This is a take-home exam
2. This exam is open book, open Internet and open MATLAB.

1. Consider a following system of linear equations:

$$3x_1 + 5x_2 - 3x_3 + 2x_4 - 8 = 0$$
$$2x_2 + 7x_3 - 4x_4 + 7 = 0$$
$$-x_1 + 2x_2 + 6x_3 + 3 = 0$$

We can write this system in matrix form as $Ax = y$, where x = $[x_1; x_2; x_3; x_4]$. How should we define matrix A and vector y in MATLAB?

(a) `A = [3 5 -3 2; 0 2 7 -4; -1 2 6 0],  y = [-8; 7; 3]`

(b) `A = [3 5 -3 2; 0 2 7 -4; -1 2 6 0],  y = [-8 7 3]`

(c) `A = [3 5 -3 2; 0 2 7 -4; -1 2 6 0],  y = [8; -7; -3]`

(d) `A = [3 0 -1; 5 2 2;-3 7 6; 2 -4 0],  y = [8 -7 -3]`

(e) `A = [3 0 -1; 5 2 2;-3 7 6; 2 -4 0],  y = [8; -7; -3]`

2. Consider a system of linear equations, $Ax = y$. The system has 8 equations with 8 unknowns. Suppose we also know that

$$Rank([A, y]) = Rank(A) = 7$$

Then, the system has:

(a) No solution

(b) A unique solution

(c) Infinitely many solutions

(d) 8 solutions

(e) 7 solutions

3. If a linear system $Ax = y$ has a unique solution and $A$ is a square matrix, which of the following is/are true?

(a) `inv(A) * y` would give the unique solution of the system.

(b) `pinv(A) * y` would give the unique solution of the system.

(c) `mldivide(A, y)` would give the unique solution of the system.

(d) $det(A) \neq 0$

(e) All of the above are true.

4. 5 different linear regressions are performed on a given set of data points $x_i$ and $y_i$, $i = 1, 2...n$. Below is the plot of the original data points along with the resulting linear fits.

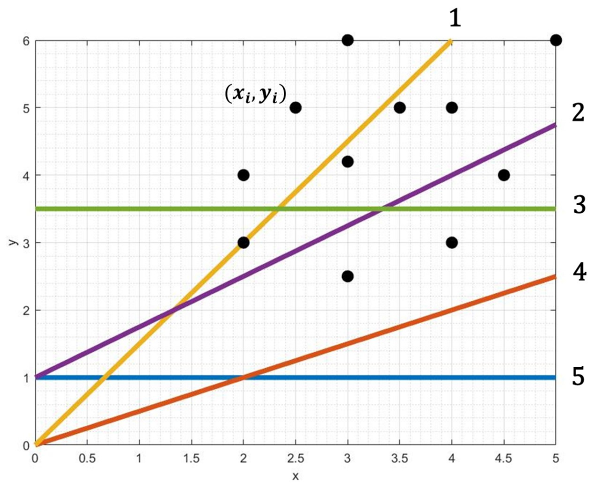



Figure 1: Original data, along with the linear fits

Recall the definition of the sum of the squared error (SSE):

$$SSE = \sum_{i=1}^{n} (\hat{y}(x_i) - y_i)^2$$

where $\hat{y}(x_i)$ are obtained from the linear fit and $y_i$ are the original data.
Which of the linear fits would give the largest $SSE$ value?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5

5. Which of the following is true about a least squares linear regression model? Recall that the residual vector is given by $r = y - \hat{y}(x)$, where $x_i$ and $y_i$, $i = 1, 2...n$, represent the data points and $\hat{y} = f(x)$ represents the model.
(Assume that we stored the residual vector in a 1D array `r` and `mean()` and `std()` are the built-in functions of MATLAB)

(a) Least squares regression minimizes $\sum_{i=1}^{n} |r_i|$.
(b) `mean(r.^2)`=0
(c) `std(r)` = 0
(d) `mean(r)` = 0
(e) None of the above is true.

6. Suppose that we are interested in estimating the spring stiffness of a linear spring, which follows the relation:

$$F = kx$$

where F is the applied load ($N$), x is the displacement ($cm$) and k is the stiffness of the spring ($N/cm$). We conducted several experiments and obtained the results shown below.

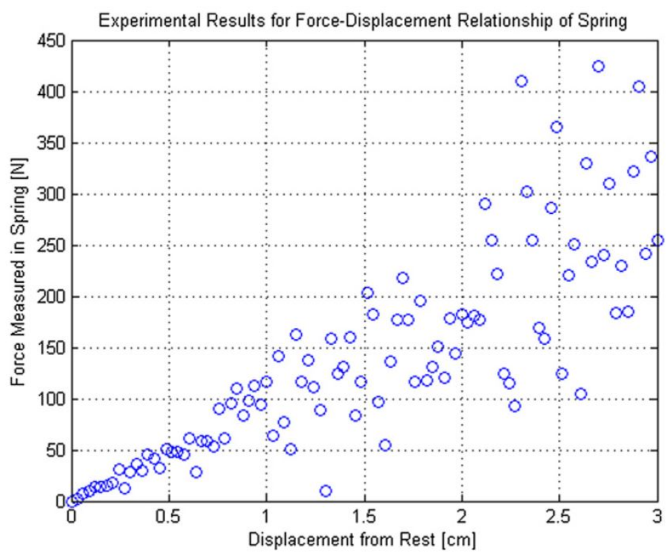



Figure 2: Each point is corresponding to a data pair $(x_i, F_i)$

Suppose that we stored all data points in row vectors `x` and `F`.

Which of the following MATLAB expressions would give the least-squares estimate for the spring stiffness k? (Hint: Your model should reflect the fact that there is no deformation if the applied load is zero.)

(a) `F\x`

(b) `inv(x * x') * x * F'`

(c) `F'\x'`

(d) `inv(x' * x) * x' * F`

(e) `F'/x'`

7. Based on least squares regression, what is the equation of the line that fits the following points: (2,5), (3,7), (4,8), (5,11)?

(a) $y = 1.9x + 1.1$

(b) $y = 1.9x - 1.8$

(c) $y = -1.8x - 1.7$

(d) $y = -1.8x + 1.7$

(e) $y = x$

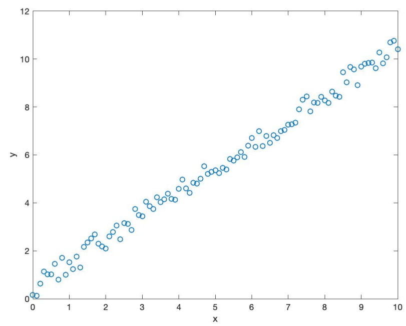
8. A unique polynomial equation is generated to pass through n+1 points. What is the maximum possible degree of this polynomial?
   (a) 0
   (b) 1
   (c) n
   (d) n+1
   (e) n+2

9. You found the best linear model $y^{ls} = a_1x + a_0$ fitting the following data using least squares regression. Given the $R^2$ formula:

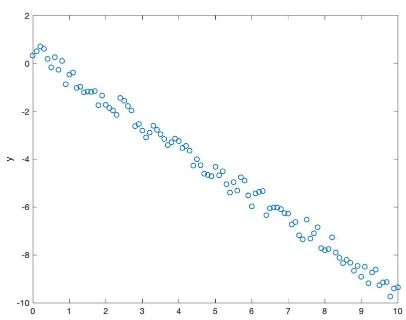
$$R^2 = \frac{\sum_{i=1}^{n}(y_i^{ls} - \bar{y})^2}{\sum_{i=1}^{n}(y_i^{data} - \bar{y})^2}$$

where $y_i^{ls}$ represents the $y$ value from linear model, $y_i^{data}$ represents the original $y$ data, and $\bar{y}$ is the mean of the original data points.
Which of the following sets of original data points $x$ and $y$ will give us smallest $R^2$ value? (Each following figure contains 100 pairs of data points)

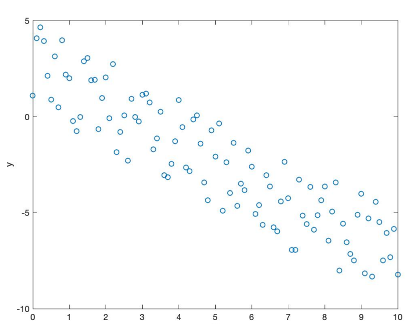
(a)



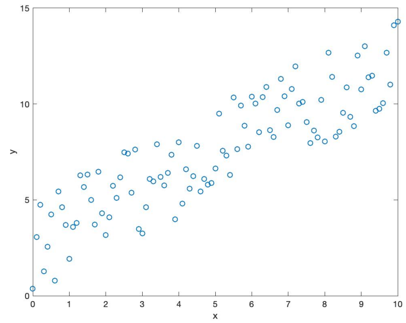

(b)



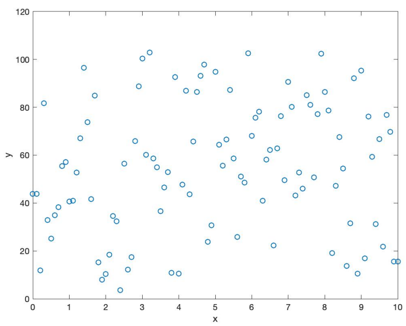

(c)



(d)




(e)



10. Given the following equation

$$0 = \sqrt{x} + x - 10$$

We have the following script to find the root of the equation:

```
fh = @(x) -x^(0.5)+10;
initialGuess=2.0;
tolerance=1e-6;
root = FixPoint(fh, initialGuess, tolerance);
```

Complete the code below so that `FixPoint(fh,initialGuess,tolerance)` can find the root. The incomplete function `FixPoint` is as follows:

```
function [fValue] = FixPoint(fh, initialGuess, tolerance)
rootValue = initialGuess;
fValue = fh(rootValue);
while(_____)
    rootValue = fValue;
    fValue = fh(rootValue);
end
end
```

Hint: We can find the root `x = 7.2984` by using the function `FixPoint`, which finds the root `x` satisfying `fh(x)=x`.

Please choose the answer that correctly completes the code:

(a) `abs(fValue - rootValue) < tolerance`

(b) `abs(fValue) > tolerance`

(c) `abs(fValue - rootValue) > tolerance`

(d) `abs(rootValue) < tolerance`

(e) `None of above`

11. We want use Newton-Raphson method to find the root of equation $3x^3 + 2x^2 + x + 9 = 0$. Please choose the answer that correctly completes the code:

```
F = @(x) 3*x^3+2*x^2+x+9;
df = @(x) 9*x^2+4*x+1;
newton_root= myNewton(F, df, 0, 1e-6);

function [newton] = myNewton(F, df, x0, tol)
  if abs(F(x0)) < tol
      newton = x0;
  else
      [newton] = myNewton(F, df,__,tol);
  end
end
```

(a) `x0`

(b) `x0+F(x0)/df(x0)`

(c) `x0-F(x0)/df(x0)`

(d) `F(x0)/df(x0)`

(e) None of above

12. We want to use Bisection method to find the root of equation $3x^3 + 2x^2 + x + 9 = 0$. Please choose the answer that correctly completes the code:

```
F = @(x) 3*x^3+2*x^2+x+9;;
bisect_root = bisection(F,-6,0,1e-6);

function  [bisect_root] = bisection(F, a, b, tol)
mid_x = (b-a)/2+a;
if abs(F(mid_x)) < tol
    bisect_root = mid_x;
elseif sign(F(a)) == sign(F(mid_x))
    bisect_root = bisection(F,__,__,tol);
elseif sign(F(b)) == sign(F(mid_x))
    bisect_root = bisection(F,__,__,tol);
end
end
```

(a) `b,mid_x; mid_x,a`

(b) `mid_x,b; a,mid_x`

(c) `a,b; a,mid_x`

(d) `mid_x,a; a,mid_x`

(e) None of above

13. You are given the following equation:

$$x^3 - 3x + 2 - e^x = 0$$

What would be the estimate of the root to the equation after conducting one step of Newton-Raphson method with an initial value of $x_0 = 0$:

(a) $-0.25$
(b) 0.25
(c) 0.2
(d) $-0.2$
(e) 0.2455

14. Say we have data points `y` (a column vector) for a time series `t` (a column vector) with length `n`. We would like to fit a 3rd order polynomial to this data. Which line of code would give us `coef` as 1x4 column vector of polynomial coefficients in decreasing order, i.e. the first element of the column vector is the coefficient which corresponds to $t^3$?

(a) `coef = [ones(n,1) t  t.^2  t.^3 ]\y`
(b) `coef = [ones(n,1)  t'  t.^2' t.^3' ]\y'`
(c) `coef = [t.^3  t.^2  t  ones(n,1)]\y'`
(d) `coef = y\[t.^3  t.^2  t  ones(n,1)]`
(e) `coef = [t.^3  t.^2  t  ones(n,1)]\y`

15. Which root finding technique is most suitable to find the root $x = 4$? The initial interval for Bisection method is [3,7] and the initial guess for Newton's method is a random point between 3 to 7.
Note: Both the technique and justification must be correct

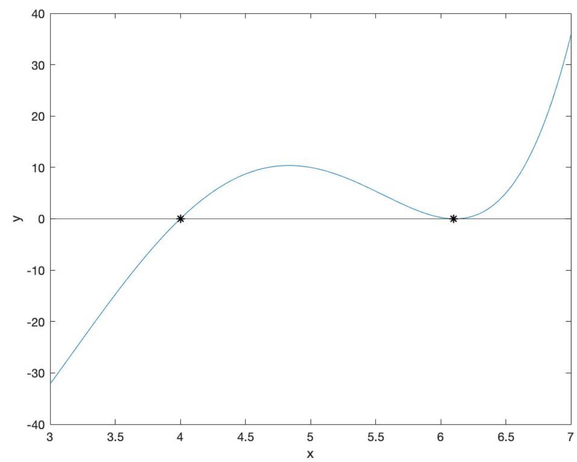



Figure 3: the plot of $x$ and $y$

(a) Bisection method : because the sign of the function value at midpoint $x = 5$ is opposite to the sign of the function value at $x = 3$.

(b) Bisection method : because all values before the root $x = 4$ are negative and all values after the root are non-negative over the range.

(c) Newton's method : because the graph is continuous

(d) Newton's method : because Newton's method converges faster than the Bisection method

(e) Newton's method : because there are multiple roots over the range

16. Which of the following functions would have the same approximation to $f'(x)$ for forward, backward, and central difference methods?

    (a) $f(x) = x^4$
    (b) $f(x) = x^3$
    (c) $f(x) = x^2$
    (d) $f(x) = x$
    (e) All of the above

17. Let $f$ be a continuous and strictly increasing function between $a$ and $b$.
    Recall from homework 8:
    Riemman left:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{N-1} hf(x_i)$$

    Riemann right:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{N} hf(x_i)$$

    Which one of the following answers is true?

    (a) Riemann left gives an overestimate of the true integral value of $f$ and Riemann right an underestimate of the true integral value of $f$
    (b) Trapezoidal rule gives an overestimate of the true integral value of $f$
    (c) Trapezoidal rule gives an underestimate of the true integral value of $f$
    (d) Riemann left gives an underestimate of the true integral value of $f$ and Riemann right an overestimate of the true integral value of $f$
    (e) None of the above

18. Consider the following piece-wise linear function. If f_der_A, c_der_A and b_der_A are the forward, centered and backward differentiation at point A, respectively, with **a step size of 1**, what is the relationship between the three?

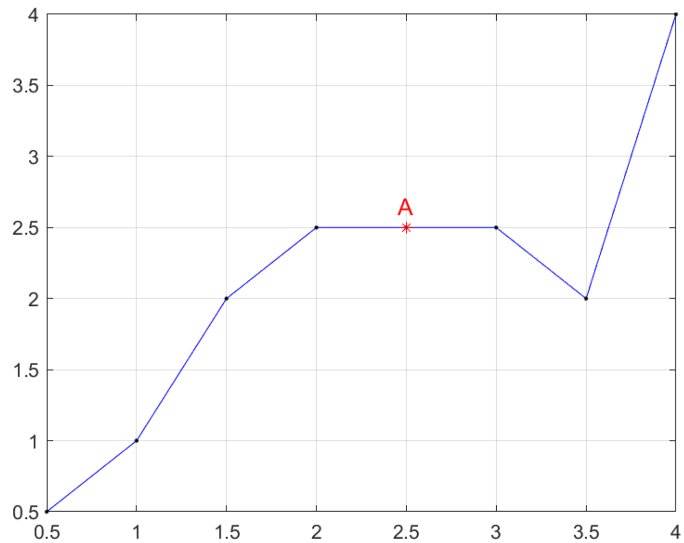



Figure 4: Piece-wise linear function

(a) b_der_A < c_der_A < f_der_A

(b) b_der_A = c_der_A = f_der_A

(c) b_der_A > c_der_A > f_der_A

(d) b_der_A > c_der_A and c_der_A < f_der_A

(e) b_der_A < c_der_A and c_der_A > f_der_A

19. The highest order of polynomials that Simpson's method can calculate the exact integral for is:

(a) 2nd order

(b) 3rd order

(c) 4th order

(d) 5th order

(e) 6th order

20. What is the order of the local error of RK4, given the step size h?
    (a) $O(h)$
    (b) $O(h^2)$
    (c) $O(h^3)$
    (d) $O(h^4)$
    (e) $O(h^5)$

21. Consider the following IVP (initial value problem):

$$\frac{df}{dt} = 0.2t$$
$$f(3) = y_0$$

Using forward Euler with a step size of 1, one calculates that

$$f(5) = 2$$

What is the value of $y_0$?

    (a) 0
    (b) 0.2
    (c) 0.6
    (d) 1
    (e) 1.6

22. Consider the ODE $y''(t) + y'(t) + y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 5$ . Which of the following plots the solution $y(t)$?
    (a) `[y, t]=ode45(@(t) [y(2); -y(2)-y(1)], [0 10], [0; 5]); plot(t, y(:, 1))`
    (b) `[t, y]=ode45(@(t, y) [y(1); -y(2)-y(1)], [0 10], [0; 5]); plot(t, y(:, 2))`
    (c) `[t, y]=ode45(@(t, y) [-y(2)-y(1); y(1)], [0 10], [0; 5]); plot(t, y(:, 1))`
    (d) `[t, y]=ode45(@(t, y) [y(2); -y(2)-y(1)], [0 10], [0; 5]); plot(t, y(:, 1))`
    (e) None of the above

23. Given the function,

```
function out = exam(f, x, tol)
out =zeros(size(x));
for i=1:length(x)
    w=1;
    temp=(f(x(i)+w)-f(x(i)-w))/(2*w);
    w=0.5;
    d=(f(x(i)+w)-f(x(i)-w))/(2*w);
    while abs(d-temp)>tol
        temp=d;
        w=w/2;
        d=(f(x(i)+w)-f(x(i)-w))/(2*w);
    end
    out(i)=d;
end
end
```

Assume that `f` is a function handle, `x` is a row vector of unique numbers, and `tol` is a positive scalar value such that $tol \ll 1$. The function `exam` is an implementation of:

(a) Interpolation
(b) Root Finding
(c) Numerical Differentiation
(d) Numerical Integration
(e) None of the above

24. Consider the following three methods of generating one million random numbers:

```
% First method
tic;
for ctr = 1:10^6
    data(ctr) = rand;
end
toc

% Second method
tic;
data=zeros(1,10^6);
for ctr = 1:10^6
    data(ctr) = rand;
end
toc

% Third method
tic;
data = rand(1,10^6);
toc
```

Which option will take the least time?

(a) First method
(b) Second method
(c) Third method
(d) They are all of the same order of magnitude (i.e., no one is significantly faster than the other two).
(e) Depends on your hardware.

25. Which one of the following statements is **False**?

(a) The time complexity of the retrieval of $i^{th}$ element from array `x` using `x(i)` is $O(1)$.
(b) In MATLAB, using the index `‘a’` is valid for a 1-by-100 array. For example, `X(‘a’)`.
(c) In MATLAB, the code `2 = x` results in an error.
(d) Preallocation of memory will always make a program slower.
(e) One byte is equal to 8 bits.