**Math 1B — UCB, Fall 2019 — M. Christ**

*Solutions for Midterm Exam 2a*

**(1)** Determine whether each of the following infinite series converges, or diverges. Show your reasoning. Justify your answers by citing theorems/tests by name.

**(1a)** $\sum_{n=1}^{\infty} n^{-3}\sin(n^2)$

**Solution.** Let $a_n = n^{-3}\sin(n^2)$ and $b_n = n^{-3}$. Then $|a_n| \le b_n$ for every index $n$.

The series with terms $b_n$ converges by the $p$-test, with $p = 3 > 1$. The series with terms $|a_n|$ therefore converges by the comparison test. Thus the given series is absolutely convergent. Therefore it converges. $\square$

**(1b)** $\sum_{n=2}^{\infty}(-1)^{n+1}n^{1/n}$

**Solution.** Let $a_n = (-1)^{n+1}n^{1/n}$. Then $|a_n| = n^{1/n}$. We have learned that $\lim_{n\to\infty} n^{1/n} = 1$. Therefore $\lim_{n\to\infty}|a_n| \ne 0$. Therefore it is not true that $\lim_{n\to\infty} a_n$ exists and is equal to zero. According to the Divergence test, the given series diverges. $\square$

**(1c)** $\sum_{n=1}^{\infty} n^{-1}\arctan(n)$

**Solution.** Use the limit comparison test, with $a_n = n^{-1}\arctan(n)$ and $b_n = n^{-1}$. The series with terms $b_n$ is the harmonic series, which we know to be divergent.

Then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\arctan(n) = \frac{\pi}{2}.$$

This limit is $> 0$, and $< \infty$. Therefore according to the limit comparison test, either both series converge, or both diverge. We know that the series with terms $b_n$ diverges, so we conclude that the given series also diverges. $\square$

**(1d)** $\sum_{k=2}^{\infty}\frac{1}{k(\ln(k))^4}$

**Solution.** Let $a_n$ be the terms of the given series. Use the integral test. Define $f(x) = \frac{1}{x(\ln(x))^4)}$. This is a positive, decreasing, continuous function on the interval $[2,\infty)$, which satisfies $f(n) = a_n$ for every positive whole number $n$.

The improper integral $\int_2^{\infty} f(x)\,dx$ converges. To see this, use the substitution $x = e^u$ to express

$$\int \frac{1}{x(\ln(x))^4}\,dx = \int u^{-4}\,du = -\tfrac{1}{3}u^{-3} + C = -\tfrac{1}{3}(\ln(x))^{-3} + C.$$

Therefore

$$\int_2^t \frac{1}{x(\ln(x))^4}\,dx = -\tfrac{1}{3}(\ln(x))^{-3}\Big|_2^t = \tfrac{1}{3}(\ln(2))^{-3} - \tfrac{1}{3}(\ln(t))^{-3}.$$

The limit of this quantity, as $t \to \infty$, exists and is equal to $\frac{1}{3}(\ln(2))^{-3}$. Therefore the improper integral converges.

By the integral test, the given series also converges. $\square$

**(2a)** Determine the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^n}{1+\ln(n)}(x-1)^n$. Show your work and reasoning.

**Solution.** The series converges for $x = 1$ since every term is then equal to zero. Assume now that $x \neq 1$. Let $a_n = \frac{(-1)^{n+1}2^n}{1+\ln(n)}(x-1)^n$. Then

$$\frac{|a_{n+1}|}{|a_n|} = 2|x-1|\frac{1+\ln(n+1)}{1+\ln(n)}.$$

To analyze the limit of $\frac{1+\ln(x+1)}{1+\ln(x)}$ as $x \to \infty$, observe that both numerator and denominator have limit $\infty$. Therefore l'Hôpital's rule can be applied:

$$\lim_{x\to\infty} \frac{1+\ln(x+1)}{1+\ln(x)} = \lim_{x\to\infty} \frac{(x+1)^{-1}}{x^{-1}} = \lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{1}{1+x^{-1}} = 1.$$

Therefore

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2|x-1|.$$

Therefore according to the ratio test, the series converges when $|x-1| < \frac{1}{2}$, and diverges when $|x-1| > \frac{1}{2}$. (The test is inconclusive when $|x-1| = 1$.) Thus the radius of convergence is equal to $\frac{1}{2}$. $\square$

**(2b)** State the Limit Comparison Test. (An accurate statement includes any hypotheses.)

**Solution.** Let $\sum a_n$ and $\sum_n b_n$ be infinite series with positive terms. Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ exists for some real number $c > 0$. Then either both series converge, or both series diverge. $\square$

*In class, I included the cases in which the limit equals* $0$ *or* $\infty$ *in the statement of the test. In our text, they are stated separately (as exercises). Therefore on this exam, full credit will be given for the version in the text.*

**(3)** Let $s$ be the sum of the series $\sum_{k=1}^{\infty} ke^{-k}$. For $n = 1, 2, 3, \ldots$ let $s_n$ be the $n$–th partial sum of this series. Show how to find $n$ sufficiently large to guarantee that $|s - s_n| < 10^{-4}$. Justify your steps.

**Solution.** The function $f(x) = xe^{-x}$ satisfies $f'(x) = (1-x)e^{-x} = -(x-1)e^{-x}$, which is $< 0$ when $x > 1$. Therefore it is decreasing on the interval $[1, \infty)$. It is also positive and continuous on that interval. Therefore the integral test applies to the given series.

According to the integral test remainder bound,

$$\int_n^\infty f(x)\,dx < s - s_n < \int_{n+1}^\infty f(x)\,dx.$$

Therefore

$$|s - s_n| < \int_n^\infty f(x)\,dx.$$

An antiderivative of $f$ is

$$F(x) = -(1+x)e^{-x}.$$

(This can be computed by integrating by parts.)

Therefore

$$\int_n^t f(x)\,dx = -(1+t)e^{-t} + (1+n)e^{-n}.$$

Consequently

$$\lim_{t\to\infty} \int_n^t f(x)\,dx = (1+n)e^{-n}.$$

Therefore our final answer is the smallest whole number $n$ that satisfies

$$(1+n)e^{-n} < 10^{-4}.$$

$\square$

**(4)** Let $f(x) = \sin(3x)$. Show that the Taylor series for $f$ at $a = \frac{\pi}{4}$ converges to $f(x)$ for every real number $x$. (In your solution, you are not permitted to use any similar results about convergence of Taylor series that we have already established for this or other functions, such as $\sin(x)$ and $\cos(x)$.)

**Solution.** For each $n$, $f^{(n)}(x)$ is one of the four functions $\pm 3^n \sin(3x)$, $\pm 3^n \cos(3x)$. Therefore

$$|f^{(n)}(x)| \le 3^n \quad \text{for every real number } x.$$

Let $R_n(x)$ be the remainder in the formula $f(x) = T_n(x) + R_n(x)$, where $f(x) = \sin(3x)$ and $T_n$ is the $n$-th degree Taylor polynomial at $a = \frac{\pi}{4}$. Apply Taylor's inequality with

$$M_n = 3^{n+1}$$

to obtain

$$|R_n(x)| \le \frac{3^{n+1}}{(n+1)!}|x-a|^{n+1} = \frac{1}{(n+1)!}|3(x-a)|^{n+1} = \frac{|y|^{n+1}}{(n+1)!}$$

with $y = 3(x-a) = 3(x - \frac{\pi}{4})$ for any real number $x$.

We have learned that $\lim_{k\to\infty} \frac{|y|^k}{k!} = 0$ for every real number $y$. Therefore $\lim_{n\to\infty} |R_n(x)| = 0$ for every $x$. Therefore $\lim_{n\to\infty} R_n(x) = 0$ likewise. Therefore the Taylor series at $a = \frac{\pi}{4}$ converges to $\sin(3x)$ for every real number $x$. $\square$

**(5)** Prove this part of the comparison test: If $0 < a_n \leq b_n$ for every $n$, and if the series $\sum_{n=1}^{\infty} b_n$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. (You are not permitted to use any part of the comparison test in this proof.)

**Solution.** (There is no need to assume that $a_n > 0$; the test applies so long as $a_n \geq 0$.)

Assume that $0 \leq a_n \leq b_n$ for every $n$, and that the series with terms $b_n$ converges. Define $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ to be the partial sums of these two series.

Because the comparison series converges, its partial sums form a bounded sequence. Thus there exists a finite number $T$ that satisfies $t_n \leq T$ for every $n$.

For every $n$,

$$s_n = a_1 + a_2 + \cdots + a_m \leq b_1 + b_2 + \cdots + b_n = t_n,$$

by basic properties of addition and inequality. Therefore $s_n \leq t_n \leq T$ for every $n$. Thus the sequence $(s_n)$ of partial sums is bounded above.

This sequence is bounded below by $a_1$ since

$$a_1 + a_2 + \cdots + a_n \geq a_1$$

since every term is $\geq 0$.

Finally, this sequence of partial sums is nondecreasing, since

$$s_{n+1} = s_n + a_{n+1} \geq s_n$$

since $a_{n+1} \geq 0$.

A fundamental theorem states that any sequence that is bounded and nondecreasing, is convergent. Therefore the sequence $(s_n)$ of partial sums of the given series is a convergent sequence. Therefore the series itself converges. $\square$

**(6)** Find all $t > 0$ such that $\sum_{k=1}^{\infty} t^{\ln(k)}$ is a convergent series. Justify your answer, showing all steps of your reasoning.

**Solution.** (This is either the easiest problem on the exam, or the hardest. I awoke on Wednesday morning with the conviction that the exam was excessively long, so deleted it.)

If $t > 1$ then $t^{\ln(k)} \to \infty$ as $k \to \infty$ (since $\ln(k) \to \infty$), so the series diverges by the Divergence test. If $t = 1$ then all terms of the series equal 1, and again it diverges, by the Divergence test.

Assume $0 < t < 1$. Express $t = e^{-p}$ with $p = \ln(1/t) > 0$. The terms of the series are then

$$t^{\ln(k)} = (e^{-p})^{\ln(k)} = e^{-p\ln(k)} = (e^{\ln(k)})^{-p} = k^{-p}$$

and thus we have a $p$-series. This series converges if and only if $p > 1$. Equivalently, it converges

$$\text{If and only if } t < e^{-1}.$$

□