Problem 1.

(12 points)

(a) Solve the given vector equation for α, or explain why no solution exists:

$$\alpha \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

A:

Performing the indicated operations we obtain the vector equations (2 points)

$$\begin{bmatrix} \alpha \\ 2\alpha \\ -\alpha \end{bmatrix} + \begin{bmatrix} 12 \\ 16 \\ 8 \end{bmatrix} = \begin{bmatrix} \alpha + 12 \\ 2\alpha + 16 \\ -\alpha + 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

Thus, if a solution α exists α must satisfy the three equations:

$$\begin{aligned} \alpha + 12 &= -1 \\ 2\alpha + 16 &= 0 \\ -\alpha + 8 &= 4 \end{aligned}$$

which leads to $\alpha = -13, \alpha = -8$ and $\alpha = 4$. since α cannot simultaneously have three different values, there is no solution to the original vector equation. (2 points)

(b) For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, compute A^2, A^3, A^4.

A: (3 points) $A^2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

(c) Find the solution set of the following linear system

$$\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 6 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 10x_2 - 10x_3 - x_4 &= 1 \end{aligned}$$

A:

The augmented matrix row-reduces to (3 points)

$$\begin{bmatrix} 1 & 0 & 1 & 4/5 & 0 \\ 0 & 1 & -1 & -1/10 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row 3 represents the equation $0 = 1$, so the original system has no solutions. The solution set is the empty set $\emptyset$. (2 points)

Problem 2.(8 points)

Define two mappings $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ 0 \end{bmatrix}, S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ xy \end{bmatrix}$$

Determine whether T, S, are linear transformations.

A:

To prove that T is a linear transformation, note that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, if we write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then we have (4 points)

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) = T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) &= \begin{bmatrix} 2\left(x_1+y_1\right)+\left(x_2+y_2\right) \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1+x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2y_1+y_2 \\ 0 \end{bmatrix} = T(\mathbf{x})+T(\mathbf{y}) \end{aligned}$$

To prove that S is not a linear transformation, observe that (2 points)

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore, (2 points)

$$\begin{aligned} S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ = S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{aligned}$$

Thus it is not the case that $S(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}) + S(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$. It follows that S cannot be a linear transformation.

Problem 3.(20 points)

True or False: If True, explain why. If False, give an explicit numerical example for which the statement does not hold.

(a) If the augmented matrix of the system $A\mathbf{x} = \mathbf{b}$ has a pivot in the last column, then the system $A\mathbf{x} = \mathbf{b}$ has no solution.

A: True. (2 points) Because there is a row of the form $[0\ 0\ \cdots\ 0 \mid b]$ with $b \neq 0$ in the augmented matrix, the linear system cannot have a solution.

(b) If for some matrix A, and some vectors $\mathbf{x}, \mathbf{b}$, we have $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the column vectors of A.

A: True. (2 points) This follows that $A\mathbf{x}$ is a linear combination of the columns of A. (3 points)

(c) Given 2×2 matrices A, B, C, if $AB = AC$ and A is not a zero matrix, then $B = C$.

A: False (2 points). Take $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $C = 0$. (3 points)

(d) If a set of n vectors in $\mathbb{R}^m$ are linearly dependent, then any vector in this set can be represented by the linear combination of other $n-1$ vectors ($n > 1$).

A: False. (2 points) Consider $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. Then $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ are linearly dependent, but $\mathbf{a}_1$ cannot be represented as the linear combination of $\mathbf{a}_2, \mathbf{a}_3$. (3 points)

Problem 4.

(10 points) For a real number c, consider the linear system

$$\begin{aligned} x_1 + x_2 + cx_3 + x_4 &= c \\ -x_2 + x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 + x_3 - x_4 &= -c \end{aligned}$$

a) For what c, does the linear system have a solution?

A:

Let us find the REF of the augmented matrix (3 points)

$$\left[\begin{array}{cccc|c} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 1 \\ 1 & 2 & 1 & -1 & -c \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1-c & -2 & -2c \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 0 & 2-c & 0 & -2c \end{array}\right]$$

Each of the two row has a pivot. Thus the linear system has a solution if and only if $c \neq 2$. (2 points)

b) Find the solution set when $c = 0$.

A: When $c = 0$, this is a homogeneous linear system. (1 points) the REF of the unaugmented matrix is (2 points)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

The free variable is x_4 and so the solution set is (2 points)

$$\left\{ \begin{bmatrix} -3x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix} \middle| x_4 \in \mathbb{R} \right\}.$$

Problem 5.

(10 points)

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the image of T contains at least two linearly independent vectors. Which of the following are a possible standard matrix of T?

$$\text{a)} \begin{bmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \qquad \text{b)} \begin{bmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 1 & -1 & -2 \end{bmatrix}$$

$$\text{c)} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{d)} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 2 & -2 \\ 2 & -1 & -1 \end{bmatrix} \quad \text{e)} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

A: First, the second column must be

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

which is satisfied by all matrices. (2 points)

Use the linearity, we find that the standard matrix A must satisfy

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is satisfied by a) c) d) but not b) e) (2 points)

The image of T, i.e. the span of the vectors in a) is

$$\left\{ c \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

which does not contain two linearly independent vectors. (3 points)

The first two columns of c) d) are already linearly independent. (3 points)

So the answer is c) d).