INDENG 172: Probability and Risk Analysis for Engineers

Fall 2020 - Midterm

Due by: 10/14 11:00 AM PST

Name: ______________________ Student ID: ____________

Instructions

This exam is open book, open notes, and open resources. However, your work must be written/typed by you and be your work alone.

The exam consists of two parts, 6 'Standard Problems' and 2 'Create Your Own Question and Solution'.

Your solutions must be uploaded to Gradescope by 10/14 11:00 AM PST. Late submissions without prior approval will not be accepted. If you have any questions or issues please post them to Piazza, using a private post if necessary.

Standard Problems

Directions

To earn full credit it is not sufficient to merely have the correct numerical answer, you must also show your work, and explain your reasoning to demonstrate that you understand the concepts being tested. Partial credit will be awarded.

Problem 1: (20 points)

Consider a fair six-sided die, with the number 1 on 2 sides, the number 2 on 3 sides, and the number 3 on 1 side. Suppose we roll the die 7 times.

1. What is the probability that we get three 1s, two 2s, and two 3s in no particular order?

2. What is the distribution of the number of 2s we could get? Include the expected value and variance.

3. Now suppose we roll the die 2 times. Let X be the sum of the 2 numbers that we rolled. Find the standard deviation of X, $Std(X)$

Problem 2: (20 points)

Suppose you have a bag of coins, containing only quarters and pennies. There are q quarters and c pennies in the bag. You draw the coins from the bag one at a time without replacement. Let X be the number of draws until you got a quarter. (e.g. If you first picked a penny and then a quarter $X = 2$).

Find the expected value of X, $E[X]$ and the variance of X, $Var(X)$ in terms of q and c. Hint: Try using indicators to solve this.

Problem 3: (15 points)

Prof. Pirutinsky purchases an unpainted 3-by-3-by-3 Rubik's Cube. He carefully paints all faces in gold. After painting, the Rubik's Cube drops to the floor and breaks apart into its 1-by-1-by-1 small cubes, with some faces painted (outer faces) and some unpainted (inner faces). Right then, you pass by and see one of the cubes roll towards you. Before you pick it up, you notice that 5 faces of the cube are unpainted. Compute the probability that the face on the bottom (you can't see) is also unpainted.

Problem 4: (15 points)

Suppose a random variable X is 4 with probability .5, -2.75 with probability .2 and 18.398 with probability .3.

1. What is the mean of X, $E[X]$?

2. What is the variance of X, $Var(X)$?

3. Graph the probability mass function of X, $p(x) = P(X = x)$

4. Roughly graph the cumulative distribution function, $F(x) = P(X \leq x)$.

5. Find the mean and variance of $Y = 10 \cdot X - 1$

Problem 5: (20 points)

Andrew (he/him), Boe (they/them), and Celine (she/her) are having a truel (laser tag). Andrew hits his target 1/3 of the time, Boe hits theirs 2/3 of the time, and Celine hits hers all of the time. They will all take round-robin turns shooting. Andrew will shoot first, then Boe, and then Celine. This order will be repeated until only one shooter (the winner) is left. If Andrew wanted to maximize his probability of winning should he shoot Celine in the first round? Why or why not? (You should make reasonable assumptions on the behavior of Andrew, Boe, and Celine)

Problem 6: (10 points)

COVID-19 testing is booming in Berkeley. At the Tang Center, on average, there are 10 people per hour who come to get tested. You may assume that people decide to arrive at the Tang Center in a way that is at most weakly dependent on each other.

1. What is the probability that exactly 10 people come to get tested in an hour?
2. You started counting people at 1:00 PM and so far 8 people have already come to get tested. What is the probability that by 2:00 PM (a total of 1 hour) there will have been exactly 10 people who have come to get tested?

Create Your Own Question and Solution

Directions For each of the following questions read the probability and application topics. You must then create your own question and solution related to the probability topic using the application topic. An example of this is given below.

The question and solution must be written by you and be uniquely yours. However, you may use the textbook, the reference material, the internet, or any other resource you want for inspiration. Make sure you cite your inspirational sources or declare that there are none. Failure to include such a statement will result in no credit. For more details see the grading information section below.

Question & Solution 1: (25 points)

Probability Topic: Anything from Chapters 1 - 4

Application Topic: Anything that interests you but no dice, coins, balls, or cards. Be creative!

Required components:

- Inspiration statement
- Question
- Solution

Question & Solution 2: (25 points)

Probability Topic: A discrete random variable from one of the families in the textbook (Bernoulli, Binomial, Poisson, Geometric, HyperGeometric, etc..)

Application Topic: Anything that interests you or even something that doesn't ;)

Required components:

- Inspiration statement
- Question
- Solution

Grading information

Standard Problem grading (similar to problem set grading, points vary)

You will be awarded points for providing the numerically correct solution, showing your work, and explaining your logic and thought process in a manner that demonstrates understanding of the basic concepts.

'Create Your Own Question and Solution' Grading (25 points each)

- Citing your inspiration or declaring that there is none is an important part of this question. No credit will be given if you fail to include a statement about inspiration.
- If you simply copy from a source, another student, or previous work of yours, you will earn 0 points. We are looking for changed details, numbers, etc..
- In addition, if your question is substantially similar to other students and you fail to cite your inspiration this may qualify as violating the honor code (read: Cheating) and the issue may be referred to the Center for Student Conduct.
- You will be awarded points for expressing a clear and unambiguous question of reasonably similar difficulty to the weekly problem sets, providing the numerically correct solution, showing your work, and explaining your logic and thought process in a manner that demonstrates understanding of the basic concepts. The exact details of the rubric used may differ from the one in the syllabus, but the overall idea is the same.

Question & Solution Example: (0 points)

Probability Topic: Expected value

Application Topic: Vaccines

Required components:

- Inspiration statement
- Question
- Solution

Inspiration: No inspiration, I have thought of this on my own.

Question: We would like to re-test the efficacy of 4 vaccines (including 1 placebo) for Fauxvid-19 in mice. Each of our 100 lab mice are randomly (and independently) given one of the four vaccines. We then expose them to Fauxvid-19 and check for active infection.

Previous research has indicated that if a mouse is given the placebo it will have an active infection with probability .9, if given Vaccine 1 it will have an active infection with probability .75, Vaccine 2 yields probability .51, and Vaccine 3 yields probability .3.

What is the expected number of mice in our lab that will have an active infection?

Solution: Let Y be a random variable denoting the number of infected mice. Let X_i be 1 if the ith mouse is infected and 0 otherwise. Then $Y = \sum_{i=1}^{100} X_i$.

$$
\begin{aligned}
E\left[Y\right] &= E\left[\sum_{i=1}^{100} X_i\right] \\
&= \sum_{i=1}^{100} E\left[X_i\right]
\end{aligned}
$$

This is by linear of expectations. Using the definition of expected value we get,

$$
\begin{aligned}
E\left[Y\right] &= \sum_{i=1}^{100} E\left[X_i\right] \\
&= \sum_{i=1}^{100} \left(1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)\right) \\
&= \sum_{i=1}^{100} \left(P(X_i = 1)\right)
\end{aligned}
$$

Since X_i is i.i.d. we can drop the i and focus on an arbitrary single mouse. Let $V_0, V_1, \ldots$ be the event that the mouse gets the placebo, vaccine 1, etc.. Then using the assumption in the question we have,

$$P(X = 1 \mid V_0) = .90$$
$$P(X = 1 \mid V_1) = .75$$
$$P(X = 1 \mid V_2) = .51$$
$$P(X = 1 \mid V_3) = .30$$

Since all the V's are mutually exclusive and their union encompasses the whole sample space (exhaustive partition), we can use the Law of Total Probability to find $P(X = 1)$ as follows,

$$\begin{aligned}
&P(X = 1) \\
&= P(X = 1 \mid V_0)P(V_0) + P(X = 1 \mid V_1)P(V_1) + P(X = 1|V_2)P(V_2) + P(X = 1 \mid V_3)P(V_3) + P(X = 1 \mid V_4)P(V_4) \\
&= .9\left(\frac{1}{4}\right) + .75\left(\frac{1}{4}\right) + .51\left(\frac{1}{4}\right) + .3\left(\frac{1}{4}\right) \\
&= 0.615
\end{aligned}$$

$P(V_0) = P(V_1) = P(V_2) = P(V_3) = P(V_4) = \frac{1}{4}$ because we assumed they are all equally likely. Thus,

$$\begin{aligned}
E[Y] &= \sum_{i=1}^{100} (P(X = 1)) \\
&= 100(.615) \\
&= 61.5
\end{aligned}$$

So the expected number of mice in our lab that will have an active infection is 61.5