Physics 7B Fall 2020 Lecture 1 Midterm 1 Solutions

Problem 1

(a)

$$\Delta E_{system} = \Delta E_{water} + \Delta E_{ice}$$ (2 points)

$\leftarrow W = 0, \ \Delta E = Q$ (2 points)

$$= m_w c_w (T_f - T_w) + m_i c_i (T_{freeze} - T_i)$$ (2 points)

$$+ m_i c_w (T_f - T_{freeze}) + m_i L$$

$$= 0$$ (2 points)

Rearranging

$$T_f = \frac{m_w c_w T_w - m_i \left[c_i (T_{freeze} - T_i) - c_w T_{freeze} + L \right]}{(m_i + m_w) c_w}$$ (1 point)

$T_{freeze} = 273 K$

$$\boxed{T_f = 299\tfrac{2}{3} \text{ K}}$$ (1 point)

(b)

$$\Delta S = \Delta S_{water} + \Delta S_{ice}$$ (2 points)

$$= \int_{T_w}^{T_f} \frac{dQ_w}{T} + \int_{T_i}^{T_{freeze}} \frac{dQ_i}{T} + \int_{T_{freeze}}^{T_f} \frac{dQ_w^{ice}}{T} + \frac{\Delta Q_{i \to w}}{T_{freeze}}$$ (2 points)

$dQ = mc\,dT$

$$\to = m_w c_w \int_{T_w}^{T_f} \frac{dT}{T} + m_i c_i \int_{T_i}^{T_{freeze}} \frac{dT}{T} + m_i c_w \int_{T_{freeze}}^{T_f} \frac{dT}{T} + \frac{m_i L}{T_{freeze}}$$ (3 points)

$$= m_w c_w \ln(T_f/T_w) + m_i \left[c_i \ln\left(\frac{T_{freeze}}{T_i}\right) + c_w \ln(T_f/T_{freeze}) + \frac{L}{273} \right]$$ (2 points)

$$\boxed{\Delta S = 164 \text{ cal/K}}$$ (1 point)

Problem 2

Consider an ideal gas of N molecules contained in a cubic room with sides of length L at temperature T and pressure P.

(a) (6 pts.) Take the average x component of a molecule's velocity to be $\bar{v}_x$. Using $\bar{v}_x$ and the quantities above, derive an expression for the frequency f with which gas molecules strikes a wall. You may also use fundamental constants.

Solution: For a single molecule, the average time between two strikes is given by

$$\Delta t = \frac{2L}{\overline{|v_x|}} \qquad \text{(3pts.)} \tag{1}$$

Then we notice that the frequency f with which gas molecules strikes a wall is related to the frequency $f_0 = 1/\Delta t$ for single molecule by

$$f = N f_0 = \frac{N}{\Delta t} = \frac{N\overline{|v_x|}}{2L} \qquad \text{(3pts.)} \tag{2}$$

Remark: Here we remark why we can use the average velocity $\overline{|v_x|}$ to compute the total frequency. The precise definition for the total frequency is

$$f = \sum_i N_i f_i = \sum_i N p(i) f_i = \sum_i \frac{N p(i)}{\Delta t_i} = \sum_i \frac{N p(i)\,|v_{i,x}|}{2L} = \frac{N\overline{|v_x|}}{2L} \tag{3}$$

where $p(i)$ is the possibility to find molecules in state i, and the average velocity is defined by $\overline{|v_x|} = \sum_i p(i)\,|v_{i,x}|$.

(b) (6 pts.) Show that the frequency can be rewritten as

$$f \approx \frac{PL^2}{\sqrt{4mkT}}$$

where m is the mass of the molecule.

Solution: To find out the frequency in terms of thermal dynamic variables, we need to rewrite $\overline{|v_x|}$ and N in terms of thermaldynamic variables. The first step is to approximate the average velocity $\overline{|v_x|}$ by the rms speed $v_{x,rms}$. We first observe

$$\frac{1}{2} m v_{rms}^2 = \frac{3}{2} k_B T \implies v_{rms} = \sqrt{\frac{3k_B T}{m}} \qquad \text{(1pts.)} \tag{4}$$

Then we can get $v_{x,rms}$ by the fact that velocity in three directions are independent and isotropic,

$$v_{rms}^2 = v_{x,rms}^2 + v_{y,rms}^2 + v_{z,rms}^2 = 3v_{x,rms}^2 \implies \overline{|v_x|} \approx v_{x,rms} = \frac{1}{\sqrt{3}} v_{rms} = \sqrt{\frac{k_B T}{m}} \qquad \text{(1pts.)} \tag{5}$$

The next step is to find N in terms of thermaldynamic variables, which can be done using the ideal gas law,

$$PV = Nk_BT \implies N = \frac{PV}{k_BT} = \frac{PL^3}{k_BT} \qquad \text{(2pts.)} \tag{6}$$

The final step is to plug everything back in to Eqn. (12),

$$f = \frac{N\overline{|v_x|}}{2L} \approx \frac{1}{2L}\frac{PL^3}{k_BT}\sqrt{\frac{k_BT}{m}} = \frac{PL^2}{\sqrt{4mk_BT}} \qquad \text{(2pts.)} \tag{7}$$

Remark: Points would not be taken off if students do not distinguish $\bar{v}_x$, $\overline{|v_x|}$, and $v_{x,rms}$. However, they have different definitions and physical meanings,

$$\bar{v}_x = \sum_i p(i)v_{i,x} = \frac{\int_{-\infty}^{\infty} v_x \exp\left(-\frac{1}{2}mv_x^2\right)\mathrm{d}v_x}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}mv_x^2\right)\mathrm{d}v_x} = 0 \tag{8}$$

$$\overline{|v_x|} = \sum_i p(i)\,|v_{i,x}| = \frac{\int_{-\infty}^{\infty} |v_x| \exp\left(-\frac{1}{2}mv_x^2\right)\mathrm{d}v_x}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}mv_x^2\right)\mathrm{d}v_x} = \sqrt{\frac{2k_BT}{\pi m}} \tag{9}$$

$$v_{x,rms} = \sqrt{\sum_i p(i)v_{i,x}^2} = \sqrt{\frac{\int_{-\infty}^{\infty} v_x^2 \exp\left(-\frac{1}{2}mv_x^2\right)\mathrm{d}v_x}{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}mv_x^2\right)\mathrm{d}v_x}} = \sqrt{\frac{k_BT}{m}} \tag{10}$$

Now we notice that $\overline{|v_x|}$ and $v_{x,rms}$ only differ by an $O(1)$ factor $\sqrt{2/\pi} = 0.80$, so the approximation we made is valid. This allows us to write down the precise formula for the total frequency,

$$f = \frac{PL^2}{\sqrt{2\pi mk_BT}} \tag{11}$$

(c) (8 pts.) Assume a cubic air-filled room is at sea level, has a temperature 20°C, and has sides of length $L = 3\text{m}$. Determine f. Assume that air is 80% Nitrogren and 20% Oxygen.

Solution: This is a tricky question as we cannot simply use the average mass for the molecules since f does not depend linear on m. The right way to carry out the computation is to use the precise definition for the total frequency,

$$f = \sum_{i\in \mathrm{N_2}} p(i)f_i + \sum_{j\in \mathrm{O_2}} p(j)f_j = \frac{N_{\mathrm{N_2}}\overline{|v_{\mathrm{N_2},x}|}}{2L} + \frac{N_{\mathrm{O_2}}\overline{|v_{\mathrm{O_2},x}|}}{2L} \tag{12}$$

$$= \frac{N_{\mathrm{N_2}}}{N}\frac{PL^2}{\sqrt{4m_{\mathrm{N_2}}k_BT}} + \frac{N_{\mathrm{O_2}}}{N}\frac{PL^2}{\sqrt{4m_{\mathrm{O_2}}k_BT}} \tag{13}$$

$$= \left(\frac{N_{\mathrm{N_2}}}{N}\frac{1}{\sqrt{m_{\mathrm{N_2}}}} + \frac{N_{\mathrm{O_2}}}{N}\frac{1}{\sqrt{m_{\mathrm{O_2}}}}\right)\frac{PL^2}{\sqrt{4k_BT}} \qquad \text{(4pts.)} \tag{14}$$

An equivalent way to understand this formula is that under thermal equilibrium, the ratio of partial pressures equals the ratio of the number of molecules,

$$\frac{P_i}{P} = \frac{N_i}{N} \tag{15}$$

Then we can use the partial pressure for N_2 and O_2 to compute their frequency separately and then sum up,

$$f = f_{N_2} + f_{O_2} = \frac{P_{N_2} L^2}{\sqrt{4 m_{N_2} k_B T}} + \frac{P_{O_2} L^2}{\sqrt{4 m_{O_2} k_B T}} = \frac{N_{N_2}}{N} \frac{P L^2}{\sqrt{4 m_{N_2} k_B T}} + \frac{N_{O_2}}{N} \frac{P L^2}{\sqrt{4 m_{O_2} k_B T}} \tag{16}$$

Now we only need to plug in the numbers $P = 1.013 \times 10^5\,\mathrm{Pa}$, $L = 3\,\mathrm{m}$, $T = 293\,\mathrm{K}$ (1 pts.) to find the final result

$$\begin{aligned} f &= \left(\frac{0.8}{\sqrt{28\,\mathrm{u}}} + \frac{0.2}{\sqrt{32\,\mathrm{u}}} \right) \frac{1.013 \times 10^5\,\mathrm{Pa} \cdot (3\,\mathrm{m})^2}{\sqrt{4 \cdot (1.66 \times 10^{-27}\,\mathrm{kg/u}) \cdot 1.38 \times 10^{-23}\,\mathrm{J/K} \cdot 293\,\mathrm{K}}} && (17) \\ &= 3.283 \times 10^{28}\,\mathrm{Hz} \qquad \text{(3pts.)} && (18) \end{aligned}$$

Remark: From Eqn. (12), we can define the effective mass

$$\frac{1}{\sqrt{m_{\text{eff}}}} = \left(\frac{N_{N_2}}{N} \frac{1}{\sqrt{m_{N_2}}} + \frac{N_{O_2}}{N} \frac{1}{\sqrt{m_{O_2}}} \right) \neq \frac{1}{\sqrt{\frac{N_{N_2}}{N} m_{N_2} + \frac{N_{O_2}}{N} m_{O_2}}} \tag{19}$$

It is important to notice that m_{eff} is not equal to the usually defined average mass $\bar{m} = \frac{N_{N_2}}{N} m_{N_2} + \frac{N_{O_2}}{N} m_{O_2}$. A common mistake is to use the average mass $\bar{m}$ instead of the effective mass m_{eff} to compute the total frequency. A similar problem can be found in Week 2 session problems regarding the rms speed of atoms in outer space.

Partial credit (4 pts.) will be given to those who use the average mass approach. In particular, two points will go to a correct average atomic mass $\bar{m} = 0.8 \cdot 28\,\mathrm{u} + 0.2 \cdot 32\,\mathrm{u} = 28.8\,\mathrm{u}$, the other two points will go to the correct numerical frequency $f = 3.279 \times 10^{28}\,\mathrm{Hz}$ using this approach. It is worth noticing that this result is the same as the correct answer up to three significant digits.

* * *

Problem 3

3. a. The power incident on the surface of the earth is equal to the solar constant multiplied by earth's cross-sectional area:

$$P = 1350 \frac{W}{m^2} \cdot \pi \times (6\times10^6 m)^2 = \boxed{1.53\times10^{17} W}$$

b. The power radiated by the earth is given by the Stefan-Boltzmann law $P = A\epsilon\sigma T^4$

for A being the surface area of the earth

$\epsilon = 1$ because the earth is a perfect emitter

$\sigma = 5.67\times10^{-8} W m^{-2} K^{-4}$ is the Stefan-Boltzmann constant

T is the surface temperature of the earth. So

$$T = \left(\frac{P}{4\pi R^2 \sigma}\right)^{\frac{1}{4}} \approx \boxed{278 K}$$

radius of earth

c. We have $\frac{dQ}{dt} = kA\frac{\Delta T}{\Delta x}$

$\frac{dQ}{dt} = 95W$ because that is the heat that needs to be dissipated. A is the surface area of the bulb, k is the thermal conductivity of glass, and $\Delta x = 0.4$ mm is the thickness.

3, c. (cont). Therefore

$$\Delta T = \frac{\Delta x}{kA}\frac{dQ}{dt} = \frac{4\times10^{-4}\,m}{0.84\,J/(s\cdot m\cdot {}^\circ C)\cdot 4\pi(0.02m)^2}(95W)$$

$$= \boxed{9.0^\circ C}$$

Problem 4

If we consider using the van der Waals equation of state for oxygen gas, experiments find that $a = 0.14 Nm^4/mol^2$ and $b = 3.2 \times 10^5 m^3/mol$ yield the best fits. Determine the pressure in 1.0 mol of the gas at 0°C if its volume is 0.70 L, calculated using

(a) (6 pts.) the van der Waals equation

Solution:

$$P = \frac{RT}{(V/n) - b} - \frac{a}{(V/n)^2} \tag{1}$$

$$P = 3242.32 kPa \tag{2}$$

(b) (6 pts.) the ideal gas law

Solution

$$P = \frac{nRT}{V} = 3242.46 kPa \tag{3}$$

Now consider a 0.5 mol sample of O2 gas that is in a large cylinder with a movable piston on one end so it can be compressed. The initial volume is large enough that there is not a significant difference between the pressure given by the ideal gas law and that given by the van der Waals equation.

(c) (8 pts.) As the gas is slowly compressed at constant temperature (300 K), at what pressure does the van der Waals equation give a volume that is 5% different than the ideal gas law volume? Use the values of a and b given above.

Solution

We need $V_{vdw}/V_{ideal} = c = 0.95$.

$$V_{vdw} = c \times V_{ideal} = c \times nRT/P \tag{4}$$

Replace V_{vdw} in the Van der Waals equation:

$$P = \frac{RT}{\frac{cRT}{P} - b} - \frac{a}{(\frac{cRT}{P})^2} = \frac{RTP}{cRT - bP} - \frac{aP^2}{(cRT)^2} \tag{5}$$

$$P = \frac{c^2(RT)^3P - caRTP^2 + abP^3}{(cRT)^2(cRT - bP)} \tag{6}$$

Dividing by P and multiplying by the denominator:

$$(cRT)^3 - (cRT)^2bP = c^2(RT)^3 - caRTP + abP^2 \tag{7}$$

Rearranging:

$$abP^2 - cRT(a - cRTb)P + (RT)^3c^2(c - 1) = 0 \tag{8}$$

Plugging in numbers and solving the quadratic equation, we get:

$$P = 5.5 \times 10^6 Pa \tag{9}$$

Problem 5

a. Along the adiabats $Q = 0$. **1**
For an isovolumetric process $W = 0$ **1**
$Q_L = \Delta E_{d\to a} = \frac{5}{2}nR(T_a - T_d)$ **2 formula + 1 sign**

b. For an isobaric process $Q_H = nC_p\Delta T$ **1**
$Q_H = n\big(\frac{d}{2}+1\big)R\Delta T$ **1**
$Q_H = \frac{7}{2}nR(T_c - T_b)$ **2 formula + 1 sign**

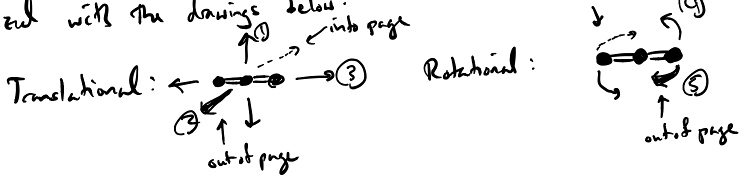
c.

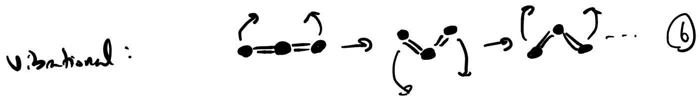
$$
\begin{aligned}
e &= \frac{W}{Q_{in}} = 1 - \left|\frac{Q_L}{Q_H}\right| && \textbf{1}\\
&= 1 - \frac{1}{\gamma}\frac{T_a - T_d}{T_c - T_b} &&\\
T_a &= T_b\left(\frac{V_b}{V_a}\right)^{\gamma-1} && \textbf{2}\\
T_d &= T_c\left(\frac{V_d}{V_c}\right)^{\gamma-1} && \textbf{2}\\
V_d &= V_a &&\\
\frac{T_c}{T_b} &= \frac{V_c}{V_b} && \textbf{1}\\
\frac{T_a}{T_d} &= \frac{T_b}{T_c}\left(\frac{V_b}{V_c}\right)^{\gamma-1} = \left(\frac{V_b}{V_c}\right)^{\gamma} && \textbf{2}\\
\Longrightarrow\quad e &= 1 - \frac{1}{\gamma}\left(\frac{V_a}{V_b}\right)^{1-\gamma}\frac{\left(\frac{V_c}{V_b}\right)^{\gamma} - 1}{\frac{V_c}{V_b} - 1} && \textbf{1}\\
\gamma &= \frac{7}{5} &&\\
e &= 1 - \frac{5}{7}\left(\frac{V_b}{V_a}\right)^{\frac{2}{5}}\frac{\left(\frac{V_c}{V_b}\right)^{\frac{7}{5}} - 1}{\frac{V_c}{V_b} - 1} && \textbf{1}
\end{aligned}
$$

Midterm 1 Bonus

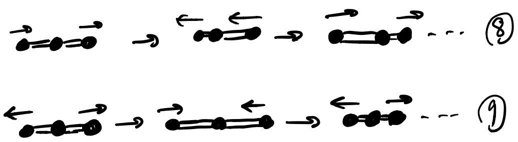
7B Fall 2020, Lecture 1

Solution: For any ideal gas, C_p in units of R is equal to $\frac{f+2}{2}$, where f is the number of degrees of freedom active at the gas's temperature. At very low temperature, CO_2 has 3 degrees of freedom from the three independent directions of translational motion, and at some temperature lower than room temperature the 2 independent rotational degrees of freedom also become available. Then, at higher temperature, the 4 modes of vibrational motion begin to unfreeze, leading to an additional 8 degrees of freedom at very high temperature. All of these degrees of freedom can be visualized with the drawings below:





Same thing but bonds vibrate into/out of page ⑦



If we look at the numbers, we need that $4.5R = \frac{f+2}{2}$, so at room temperature we effectively have that $f = 7$. This means that we have all of the translational and rotational degrees of freedom unlocked, and the lowest energy vibrational modes ⑥ and ⑦ are partially unlocked (keeping in mind that each vibrational mode counts twice in f since they contribute both kinetic and potential energy). If ⑥ and ⑦ are "half" unlocked, then the total count for f would be 3 translational + 2 rotational + $\frac{1}{2} \cdot 2 \cdot 2$ vibrational = 7 degrees of freedom, as required. **Note:** ⑥ and ⑦ unlock at the same time since they have the same energy, so we can't have ⑥ or ⑦ individually unlocked completely, hence why I say they are both "half" unlocked.

Rubric: 2 points: Use some form of $C_p = \frac{f+2}{2}R$

1 point: Mention the 3 translational degrees of freedom

1 point: Mention the 2 rotational degrees of freedom

1 point: Mention the 2 vibrational degrees of freedom

Partial Credit : 1 point: Mistakenly use $C_p = C_v = \frac{f}{2}R$

Note that a completely correct solution does not need all that I wrote above for my solution, just what is mentioned in the rubric.