S20 PHYSICS 7B: Wang Final Solutions

Friendly neighborhood 7B GSIs

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1 Problem 1

(a)

Kirchhoff's loop law gives the equation:

$$IR + \frac{Q}{C} = 0, \tag{1.1}$$

and taking $I = \dot{Q}$ gives the equation

$$\dot{Q} + \frac{1}{RC}Q = 0, \tag{1.2}$$

which is solved by

$$Q(t) = q_0 e^{-t/RC}, \tag{1.3}$$

where we used $Q(0) = q_0$.

(b)

We have

$$I = \dot{Q} = -\frac{q_0}{RC}e^{-t/RC}. \tag{1.4}$$

The power dissipated is then

$$P = I^2 R = \frac{q_0^2}{RC^2}e^{-2t/RC}. \tag{1.5}$$

(c)

By energy conservation, the power dissipated in the resistor will heat up the gas. After all the charge has been released, the total energy dissipated will be

$$\Delta E = \int_0^\infty P\,dt = \frac{q_0^2}{2C}, \tag{1.6}$$

which is nothing more than the total energy originally contained in the capacitor. All of this goes to heat up the gas:

$$Q = nC_P \Delta T = \frac{7}{2}nR(T_f - T_0). \tag{1.7}$$

We can determine nR as $nR = \frac{P_0 V_0}{T_0} = \frac{P_0 LA}{T_0}$, so

$$\frac{q_0^2}{2C} = \frac{7}{2}\frac{P_0 LA}{T_0}(T_f - T_0), \tag{1.8}$$

for which we can solve

$$T_f = \left(1 + \frac{q_0^2}{7CP_0 LA}\right)T_0. \tag{1.9}$$

The volume is then

$$V_f = \frac{nRT_f}{P_0} = LA\left(1 + \frac{q_0^2}{7CP_0 LA}\right) \tag{1.10}$$

2 Problem 2

(a)

The particle is negatively charged – the trajectory made by the particle is one for a negative charge moving to the right in a magnetic field into the page.

(b)

For motion at constant velocity, the magnetic and electric forces are in equilibrium. We must therefore have

$$F_B = qv_0B = qE, \tag{2.1}$$

which gives

$$v_0 = \frac{E}{B}. \tag{2.2}$$

(c)

The circular motion of the particle must obey

$$F = m\frac{v_0^2}{R}, \tag{2.3}$$

and the source of the force F is the magnetic field. We therefore have

$$F_B = qv_0B = m\frac{v_0^2}{R}, \tag{2.4}$$

which we can rearrange to get

$$\frac{q}{m} = \frac{v_0}{RB} = \frac{E}{RB^2}. \tag{2.5}$$

3 Problem 3

(a)

The total impedance of the circuit is given by:

$$Z_{RLC} = \frac{1}{i\omega C} + i\omega L + R, \tag{3.1}$$

which has magnitude

$$|Z_{RLC}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \tag{3.2}$$

Now we must have

$$|V_{\text{in}}(t)| = |I(t)||Z_{RLC}|. \tag{3.3}$$

For V_{out}, we see that it measures the voltage drop across the capacitor and inductor, corresponding to impedance

$$|Z_{LC}| = \left|\omega L - \frac{1}{\omega C}\right|, \tag{3.4}$$

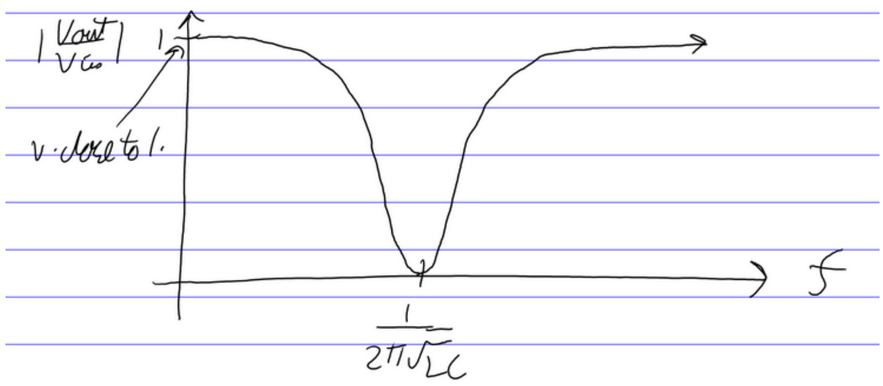
and we must have

$$|V_{\text{out}}(t)| = |I(t)||Z_{LC}|. \tag{3.5}$$

Note in particular that the circuit is a series one, and hence $I(t)$ is the same through all components. Then we immediately have

$$\frac{|V_{\text{out}}(t)|}{|V_{\text{in}}(t)|} = \frac{|Z_{RLC}|}{|Z_{LC}|} = \frac{\left|\omega L - \frac{1}{\omega C}\right|}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} = \frac{\left|\omega^2 CL - 1\right|}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2}} = \frac{\left|4\pi^2 f^2 CL - 1\right|}{\sqrt{4\pi^2 f^2 C^2 R^2 + (4\pi^2 f^2 CL - 1)^2}}. \tag{3.6}$$

When plotted as a function of f:



(b)

We see from the graph/function that the amplitude ratio vanishes for $\omega = 2\pi f = \frac{1}{\sqrt{LC}}$. Then we want

$$f = \frac{1}{2\pi\sqrt{LC}}, \tag{3.7}$$

so if we fix f, then we get

$$C = \frac{1}{4\pi^2 f^2 L} = \frac{1}{4\pi^2 (60\text{ Hz})^2 (0.1\text{ H})} \sim 70\ \mu\text{F}. \tag{3.8}$$

(c)

We want

$$\frac{1}{2} = \frac{\left|\omega^2 CL - 1\right|}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2}}, \tag{3.9}$$

which we see occurs when

$$\sqrt{\omega^2 C^2 R^2 + (\omega^2 CL - 1)^2} = 2\left|\omega^2 CL - 1\right|, \tag{3.10}$$

which corresponds to

$$\omega^2 C^2 R^2 = 3(\omega^2 CL - 1)^2, \tag{3.11}$$

which we can solve as

$$R = \frac{\sqrt{3}|\omega^2 CL - 1|}{\omega C}. \tag{3.12}$$

Plugging in numbers:

$$R \sim 10\ \Omega. \tag{3.13}$$

4 Problem 4

Let us first consider two concentric loops of wire, one of radius R_1 and one of radius R_2. The loop with radius R_1 has a CCW current I and the loop with radius R_2 has CW current I. The first loop of radius R_1 generates a magnetic field at its center pointing out of the page of strength

$$B_1 = \frac{\mu_0 I}{2R_1}. \tag{4.1}$$

The second loop generates a magnetic field at its center pointing into the page of strength

$$B_2 = \frac{\mu_0 I}{2R_2}. \tag{4.2}$$

Therefore the magnetic field at the center of the two wires will be

$$B_{12} = \frac{\mu_0 I}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{4.3}$$

pointing out of the page.

Now, we consider only the section of the loops within the angle θ, as in the figure. Clearly the parts of the loop that point away from C do not contribute any magnetic field to C. Then the the field generated at C will simply be the proportional amount according to θ, i.e.

$$B = \frac{\mu_0 I}{2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\frac{\theta}{2\pi}, \tag{4.4}$$

pointing out of the page.

5 Problem 5

(a)

The plane contributes a field $\vec{E}_1 = -\frac{\sigma}{2\epsilon_0}\hat{x}$. For the slab, we consider it as stacking together several planes with surface charge $\sigma = \rho\, dx$, which each contribute a field $dE_2 = \frac{\sigma}{2\epsilon_0} = \frac{\rho\, dx}{2\epsilon_0}$. Then the total contribution of the slab is

$$E_2 = -\int_{-d/2}^{d/2} \frac{\rho\, dx}{2\epsilon_0}\hat{x} = -\frac{\rho d}{2\epsilon_0}\hat{x}. \tag{5.1}$$

Then the field to the left of the plane is

$$E(x < -d/2) = -\left(\frac{\sigma}{2\epsilon_0} + \frac{\rho d}{2\epsilon_0}\right)\hat{x}. \tag{5.2}$$

(b)

The analysis for the right side of the slab is identical to the one above, as there is no dependence on distance outside of the slab, so

$$E(x > d/2) = \left(\frac{\sigma}{2\epsilon_0} + \frac{\rho d}{2\epsilon_0}\right)\hat{x}. \tag{5.3}$$

(c)

The thin plane contributes a field $\vec{E}_1 = \frac{\sigma}{2\epsilon_0}\hat{x}$ everywhere inside the slab. Consider a point x_0 inside the slab. There are 2 competing fields: the part of the slab with $x < x_0$ and the part with $x > x_0$. The part to the left contributes a field:

$$\vec{E}_L = \int_{-d/2}^{x_0} \frac{\rho\, dx}{2\epsilon_0}\hat{x} = \frac{\rho}{2\epsilon_0}\left(x_0 + \frac{d}{2}\right)\hat{x}, \tag{5.4}$$

while the part to the right contributes

$$\vec{E}_R = -\int_{x_0}^{d/2} \frac{\rho\, dx}{2\epsilon_0}\hat{x} = \frac{\rho}{2\epsilon_0}\left(x_0 - \frac{d}{2}\right)\hat{x}. \tag{5.5}$$

Adding everything together:

$$\vec{E}(x) = \left(\frac{\sigma}{2\epsilon_0} + \frac{\rho x}{\epsilon_0}\right)\hat{x}. \tag{5.6}$$

6 Problem 6

(a)

Let z be the axis pointing out of the page, so $\vec{B} = -B\hat{z}$. Then x points to the right and y points up. A charge q at distance r from the pivot moves with velocity:

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega\hat{z} \times r(\cos\theta\hat{x} + \sin\theta\hat{y}) = \omega r(-\sin\theta\hat{x} + \cos\theta\hat{y}). \tag{6.1}$$

Then the magnetic force is

$$\vec{F} = q\vec{v} \times \vec{B} = q\omega r B(\sin\theta\hat{x} - \cos\theta\hat{y}) \times \hat{z} = -q\omega r B(\cos\theta\hat{x} + \sin\theta\hat{y}) = -q\omega r B\hat{r}. \tag{6.2}$$

(b)

The build up of charges due to the magnetic field will generate an electric field that will eventually stabilize charges against the electric field. Consider a charge q after this equilibrium has been achieved. In equilibrium, $\vec{F}_E + \vec{F}_B = 0$, so

$$\vec{F}_E = q\vec{E} = q\omega r B\hat{r}. \tag{6.3}$$

We hence have

$$\vec{E} = \omega r B\hat{r}. \tag{6.4}$$

Integrating to find the potential difference between the ends of the rod:

$$\Delta V = -\int_0^L \omega r B\, dr = -\frac{1}{2}\omega B L^2. \tag{6.5}$$

In fact, there is a subtlety in the problem that we have not considered – there is a centrifugal force arising from the rotation of the rod! We implicitly assumed it was negligible and hence ignored it, but a more careful solution would need to consider such effects.